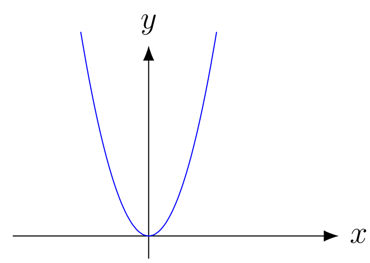
Math 111 lecture for Wednesday, Week 1

Slopes. Let $f(t) = t^2$ be a function describing the position of a particle on the real number line at time t. For instance, at time $t = 2$, the particle is at $f(2) = 4$.



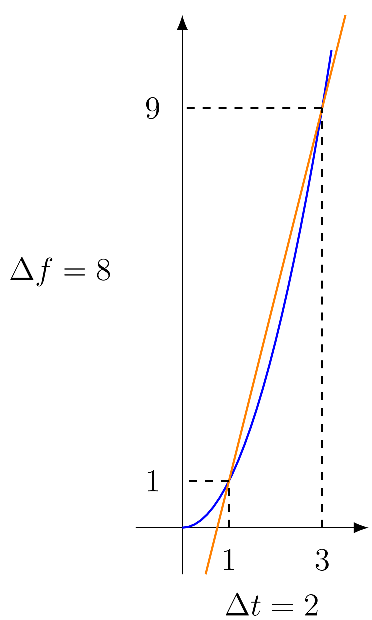
Graph of $f(t) = t^2$.

Problem: Find the average speed of the particle from time $t = 1$ to time $t = 3$.

Solution: The net distance traveled is $f(3) - f(1) = 3^2 - 1^2 = 8$, and the time elapsed is $3 - 1 = 2$. So

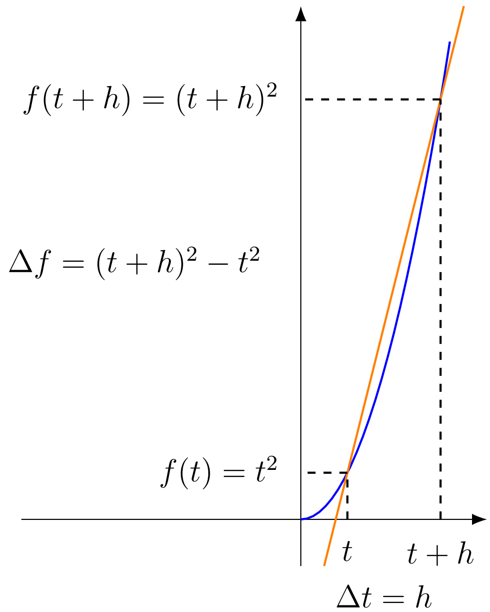
$$\text{average speed} = \frac{\Delta f}{\Delta t} = \frac{f(3) - f(1)}{3 - 1} = \frac{8}{2} = 4.$$

Relevant picture:



Graph of $f(t) = t^2$ and a secant line.

Problem: Find the average speed from an arbitrary time t to time $t+h$ for some $h > 0$.



Graph of $f(t) = t^2$ and a secant line.

We have

$$\begin{aligned}
\text{average speed} &= \frac{\Delta f}{\Delta t} \\
&= \frac{(t+h)^2 - t^2}{(t+h) - t} \\
&= \frac{t^2 + 2th + h^2 - t^2}{h} \\
&= \frac{2th + h^2}{h} \\
&= \frac{(2t+h)h}{h} \\
&= 2t + h.
\end{aligned}$$

(In the last step, we used the fact that $h \neq 0$.) For example, the average speed from time 1 to time $1.1 = 1 + 0.1$ (so $t = 1$ and $h = 0.1$) is

$$2 \cdot 1 + 0.1 = 2.1.$$

Let h get really small to estimate the *instantaneous* speed at time t: as the time interval $h = (t+h) - t$ approaches 0, the average speed approaches $2t$. We've just calculated the *derivative*, $f'(t)$, of f at an arbitrary time t and found that $f'(t) = 2t$.

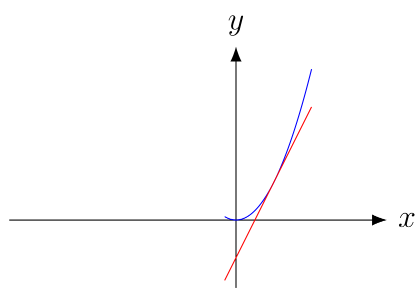
At the time $t = 1$, the instantaneous speed is $f'(1) = 2 \cdot 1 = 2$. The *best linear approximation* to the function f at time $t = 1$, i.e., the line that best describes the function f at time $t = 1$, is the line with slope $f'(1) = 2$ and passing through the point in question, $(1, f(1)) = (1, 1^2) = (1, 1)$. So the equation for the line has the form $y = mx + b = 2x + b$. To find b, we plug in the point $(1, 2)$:

$$1 = 2 \cdot 1 + b \quad \Rightarrow \quad b = 1 - 2 = -1.$$

So the best linear approximation to f at $t = 1$ is

$$y = 2x - 1.$$

The best linear approximation is also called a *tangent line.*



Graph of $f(t) = t^2$ with attached tangent line at $t = 1$.

Limits. Above, we just considered the behavior of the quotient

$$\frac{\Delta f}{\Delta t} = \frac{f(t+h) - f(t)}{h}$$

as the time interval, h, tends to 0, and we thought of t as being fixed. So, we are really considering a function of h:

$$g(h) = \frac{f(t+h) - f(t)}{h}.$$

The interesting thing about this function $g(h)$ is that it is *not defined* at the point we are interested in, i.e., at $h = 0$. This will *always* happen when we are interested in calculating an instantaneous speed. *Our task, though, is to find out what $g(h)$ gets close to as h gets close to 0.*

If f is a really nice function—like most of those with which you are familiar, and unlike the function g, above—the question "What value does $f(x)$ get close to as x gets close to 0?" is trivial to answer. For instance, if $f(x) = x^2 + 2x + 7$, then $f(x)$ gets close to 7 as x gets close to 0. However, we have just seen that when trying to find the instantaneous change of a function, we must consider functions that are not quite so nice. For instance, the function g, above, is not even defined when $x = 0$. So our **first task** in this class is to answer the

> **Main question:** What value does $f(x)$ get close to as x gets close to some given number?

Here are some problems that reveal some of the intricacies involved in answering the question.

Problem: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 1/10^n \text{ for } n = 1, 2, \ldots \\ 3 & \text{otherwise.} \end{cases}$$

Is it true that $f(x)$ gets close to 3 as x gets close to 0? Is it true that $\lim_{x\to 0} f(x) = 3$?

Problem: Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Is it true that as $f(x)$ gets close to 1 as x get close to 0? Gets close to -1? Is it true that $\lim_{x\to 0} f(x) = 1$ or $\lim_{x\to 0} f(x) = -1$.

At this point, all that is expected is to see that the answer to the question is more complicated than one might at first suppose. In fact, the following definition precisely captures the idea we need, no more and no less. It is quite complicated. So we will need to take some time over the next few lectures to unravel it. It is included here just to indicate the task ahead of us.

> **Definition.** Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of $f(x)$ as x approaches c* is L, denoted $\lim_{x\to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x satisfies
>
> $$0 < |x - c| < \delta$$
>
> then
>
> $$|f(x) - L| < \varepsilon.$$