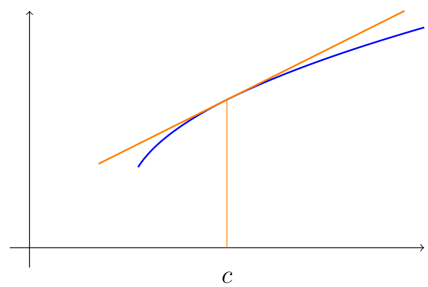
Math 111 lecture for Monday, Week 1

Overview of Calculus

The main idea of calculus is to approximate curvy things with straight things. It applies this idea to two seemingly unrelated topics: rates of change and areas.

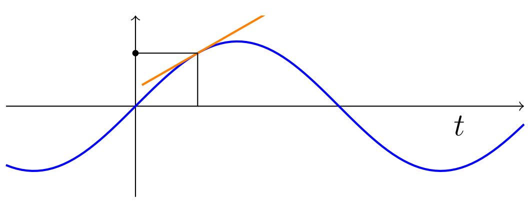
Derivatives. Let g be a function: for each real number x, the function gives a corresponding number $g(x)$. At each real number c, calculus finds the "best" line approximating the function:



Graph of a function g and its best linear approximation at the point c.

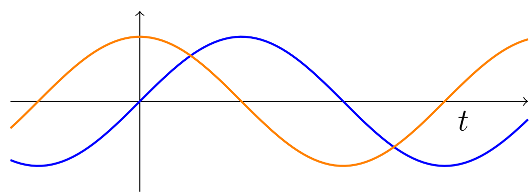
The slope of the best approximating line at a point c is the *rate of change* or *slope* of the function at that point. In the graph above, you can see that the slope of g changes from point to point: it is decreasing as c increases.

If you think of $g(t)$ as specifying the distance of a point on the y-axis from the origin at time t, then the derivative of g at time $t = c$, denoted $g'(c)$, is the *speed* the particle is traveling. For instance, suppose $g(t) = \sin(t)$. The graph appears below:



Graph of $g(t) = \sin(t)$.

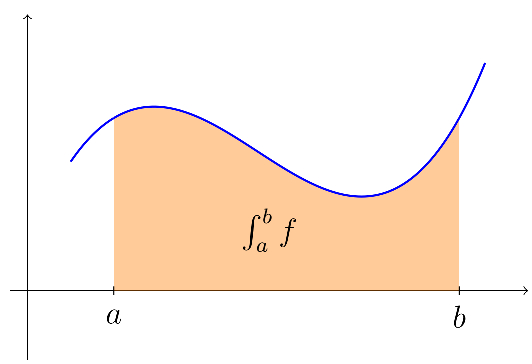
Imagine what happens to the point on the y-axis as time moves forward: it oscillates between -1 and 1. At the point pictured above, the derivative of g, i.e., the speed of the particle, is positive—the point is moving away from the origin. The derivative of $\sin(t)$ happens to be $\cos(t)$. The next picture shows the graphs of these functions superimposed.



Graph of $\sin(t)$ in blue and its derivative $\cos(t)$ in orange.

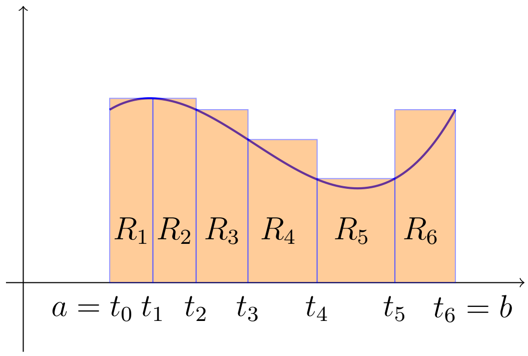
The height of the orange graph gives the slope of the blue graph at each point. For instance, notice that where the orange graph is 0 (where it hits the x-axis) the blue graph flattens out. Where the orange graph is positive, the blue graph is increasing—it's slope is positive, and where the orange graph is negative, the blue graph is decreasing.

Integrals. The integral of a function $f(t)$ from $t = a$ to $t = b$ is the area under the graph of f between those two points:



The integral $\displaystyle\int_a^b f$ is the area under the curve from $t = a$ to $t = b$.

The approach of calculus to find this area is to first estimate it using rectangles (replacing curvy things by straight things):

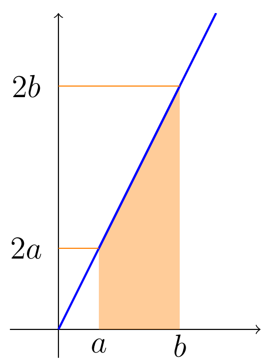


Approximating the area under the graph of f with rectangles.

The Fundamental Theorem of Calculus. It turns out there is an essential connection between derivatives and integrals. To fix ideas, let g be a distance function for a particle on the y-axis, and at each time t, let $f(t)$ be the speed of the particle. So $g'(t) = f(t)$. The Fundamental Theorem of Calculus says the following:

$$\underbrace{\int_a^b f}_{\text{area under } f \text{ from } a \text{ to } b} = \int_a^b g' = \underbrace{g(b) - g(a)}_{\text{net change in } g}.$$

Example of the Fundamental Theorem. Let $g(t) = t^2$. It turns out that $g'(t) = 2t$. Then $\int_a^b g'$ is the area pictured below:



$\int_a^b g'$ is the area under the graph of $g'(a)$ from a to b.

The area under g' from a to b can be calculated by subtracting the areas of two triangles: the one from the origin out to the line at $t = b$ minus the one from the origin out to the line $t = a$:

$$\int_a^b g' = \frac{1}{2}\, b(2b) - \frac{1}{2}\, a(2a) = b^2 - a^2.$$

Recall that $g(t) = t^2$. So in accordance with the Fundamental Theorem of Calculus, we have

$$\int_a^b g' = g(b) - g(a).$$

Summary of goals for Math 111

- What is speed? (derivatives)
- What is area? (integrals)
- How are they related? (Fundamental Theorem of Calculus (FTC))
- Theory:
  - IVT (intermediate value theorem)
  - EVT (extreme value theorem)
  - MVT (mean value theorem)
  - Chain rule, product rule
  - FTC.
- Applications:
  - Calculate speed and area efficiently.
  - Optimization (maximize and minimize functions).
  - Related rates.
  - Differential equations and population models.

The first technical definition we'll need to come to terms with is the following (to be taken up during the next lecture):

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of $f(x)$ as x approaches c* is L, denoted $\lim_{x\to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta$$

implies

$$|f(x) - L| < \varepsilon.$$