Math 111 lecture for Friday, Week 1

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of $f(x)$ as x approaches c* is L, denoted $\lim_{x\to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta$$

implies

$$|f(x) - L| < \varepsilon.$$

Comments.

- The definition of the limit contains a huge amount of information. Unless you have worked with it before—which I am not assuming—don't expect to understand it on first reading (or on the second or third, for that matter).

- We are interested in the behavior of the function f near the point c, but not exactly at the point c. In fact, f need not even be defined at c. For example, consider the function

  $$f(x) = \frac{x^2 - x}{x}.$$

  If we try to evaluate f at 0, we get $f(0) = \frac{0}{0}$, which does not makes sense (you can't divide by 0), i.e., $\frac{0}{0}$ is not a number.

- When you see the absolute values in the definition, you should think "distance". The *distance* between real numbers a and b is $|a - b|$. So you should translate $|f(x) - L| < \varepsilon$ as "the distance between $f(x)$ and the number L is less than ε".

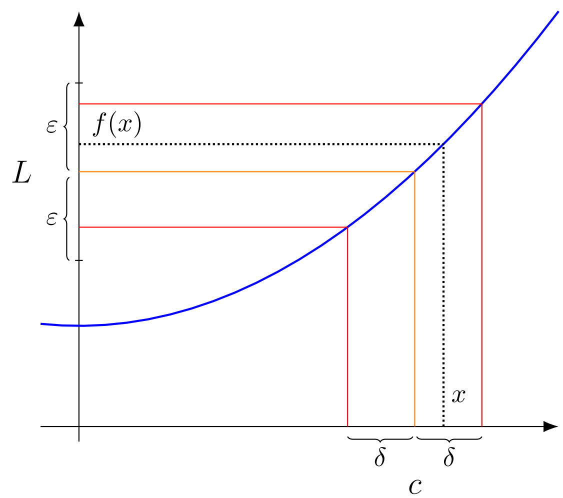
- Consider the part of the definition that says $0 < |x - c| < \delta$. If the expression had just been $|x - c| < \delta$, the requirement would be that the distance between the number x and c is less than δ. What about the fact the $0 < |x - c|$? The only way the absolute value of a number such as $x - c$ can be 0 is if the number itself is 0, i.e., $x - c = 0$ or, equivalently, $x = c$. Thus, requiring $0 < |x - c|$ is just requiring that x not equal c. This is just what we need since, after all, the function f may not be defined at c.

- Note the *quantifiers* "for all" and "there exists" in the definition. It takes a while to appreciate their importance, but they are essential. First take the "for all" part. The definition say that for all $\varepsilon > 0$, we are going to want $|f(x) - L| < \varepsilon$. Translating: for all $\varepsilon > 0$, we will want to make the distance between $f(x)$ and L less that $\varepsilon > 0$. Our goal is to make f close to L, and the ε is a measure of how close. By making ε small and requiring $|f(x) - L| < \varepsilon$, we are ensuring that $f(x)$ is within a distance of ε from L.

Next, consider the "there exists" part of the definition. It says that if you want $f(x)$ to be with a distance of ε of L, then you need to make $0 < |x - c| < \delta$. In other words, you need to make x within a distance of δ of c (remembering that we don't care what happens when $x = c$).

Given any $\varepsilon > 0$ (a challenge to make $f(x)$ close to L), you want to find an appropriate distance $\delta > 0$ (so that if x is δ-close to c, then $f(x)$ is ε-close to L).

- The relevant picture:

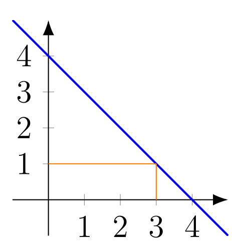


Meaning of ε and δ in the definition of the limit.

If f is steep near c, then δ needs to be taken smaller.

Problems: Here are some examples with pictures (but without proofs).

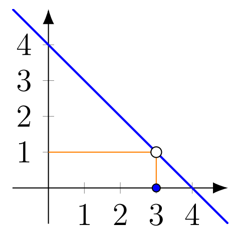
I. $\lim_{x \to 3} 4 - x = 1$.



II.

$$f(x) = \begin{cases} 4 - x & \text{if } x \neq 3 \\ 0 & \text{if } x = 3. \end{cases}$$

Here, $\lim_{x \to 3} f(x) = 1$, again. The limit would be the same even if f were not defined at all at $x = 3$.

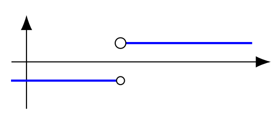


III.

$$f(x) = \frac{|x - 5]}{x - 5}.$$

Here, f is not defined at $x = 5$. However, in fact, we have

$$f(x) = \frac{|x - 5]}{x - 5} = \begin{cases} 1 & \text{if } x > 5 \\ -1 & \text{if } x < 5 \\ \text{undefined} & \text{if } x = 5. \end{cases}$$



In this case, f does not have a limit as x approaches 5. This makes sense: if x approaches 5 from numbers greater than 5, then $f(x)$ gets close to 1 (and in fact is exactly equal to 1 for all numbers greater than 5). However, the limit can't be 1 since there will be numbers x, equally close to 5 but to the left of 5 for which $f(x) = -1$. (Here, there is no way we will be able to "beat" and ε that is less than or equal to $\frac{1}{2}$ since requiring $|f(x) - L| < \frac{1}{2}$, i.e., strictly less than $\frac{1}{2}$, will be bad news if f varies from 1 to -1 when close to $x = 5$.

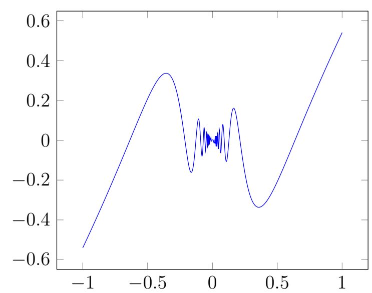
Problem. Prove that $\lim_{x \to 3} 2x + 5 = 11$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/2$. Suppose that $0 < |x - 3| < \delta$; in other words, suppose that $0 < |x - 3| < \varepsilon/2$. Then

$$\begin{aligned} |(2x+5) - 11| &= |2x - 6| \\ &= |2(x-3)| \\ &= 2|x-3| \\ &< 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Problem. Prove that $\lim_{x \to 0} x \cos(1/x) = 0$.



Graph of $f(x) = x\cos(1/x)$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Suppose that $0 < |x - 0| < \delta$; in other words, suppose that $0 < |x| < \varepsilon$. Then, since $|\cos(y)| \leq 1$ for all y, we have

$$\begin{aligned} |x\cos(1/x) - 0| &= |x||\cos(1/x)| \\ &< |x| \\ &< \varepsilon. \end{aligned}$$

□