Math 111 Homework for Wednesday, Week 14

PROBLEM 1. Let $y(t)$ represent the size of a population varying over time. In class, we saw that if $y(t)$ satisfied the differential equation

$$y'(t) = r(S - y(t))$$

for some positive constants r and S, then

$$y(t) = S - (S - I)e^{-rt}$$

where $I = y(0)$, the initial population. Thus, over time the population converges exponentially quickly to S with the rate of convergence governed by the number r.

Suppose instead that $y(t)$ satisfies

$$y'(t) = rt(S - y(t)). \tag{1}$$

This model is similar to the first one but now the rate of convergence is governed by rt as opposed to r.

(a) Find the most general solution $y(t)$ to equation (1).

(b) What is the limit, $\lim_{t\to\infty} y(t)$?

(c) Find the particular solution when the initial population is $2S$.

(d) Find the particular solution when the initial population is $S/2$.

PROBLEM 2. Suppose that a swimming pool contains 13,000 gallons of water and 350 pounds of salt. As it rains over the winter, water is drained from the pool at a rate of 225 gallons per week to keep the water level constant. How much salt is in the pool after 7 months (28 weeks)?

The difficulty with this question is that as the water is drained and replaced by rainwater, the salt concentration is decreasing. We can use a differential equation to solve the problem. Le $s(t)$ denote the number of pounds of salt in the pool at time t (with t measured in weeks).

(a) In terms of $s(t)$, what is the concentration of salt in the pool at time t (in pounds/gallon)?

(b) We have

$$s'(t) = -\text{ (number of pounds of salt lost per week)}.$$

Write a differential equation for $s'(t)$ in terms of $s(t)$.

(c) Solve the differential equation to find $s(t)$ with initial condition $s(0) =$ 350 pounds.

(d) How many pounds of salt are lost after 28 weeks? Use a calculator to give an approximate solution. (This is the amount I will have to add at the beginning of the swimming season.)

Problem 3. (Review of the proof of the fundamental theorem of calculus.) Before beginning this problem, please carefully re-read the lecture notes for class on Wednesday, Week 12. Start by reviewing the definition of the definite integral and the statement of the mean value theorem appearing in the Addendum to those notes.

Let $f(x) = 2x$ and $g(x) = x^2$ so that $g' = f$. Define the partition

$$P = \{0, 1, 2, 3, 4\}.$$

(a) Complete the following table. The column labeled c_i should be filled with a point c_i in the interval $[t_{i-1}, t_i]$ such that

$$g'(c_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$

The mean value theorem guarantees the existence of c_i.

subintervals	$\frac{g(t_i)-g(t_{i-1})}{t_i-t_{i-1}}$	c_i	$\text{glb}(f([t_{i-1}, t_i]))$	$\text{lub}(f([t_{i-1}, t_i]))$
$[0, 1]$	1	$\frac{1}{2}$	0	2

(b) Compute $L(f, P)$ and $U(f, P)$.

(c) Compute $\sum_{i=0}^{4}(g(t_i) - g(t_{i-1}))$. (By necessity, its value is between that of the lower and upper sums.)