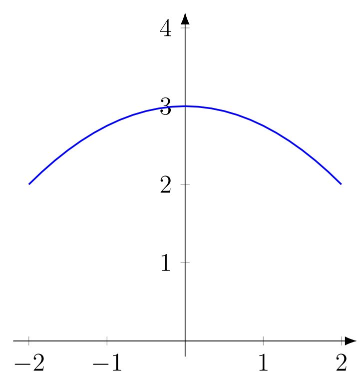
Math 111 Homework for Friday, Week 9

The purpose of this homework is two-fold. First, it provides practice computing upper and lower sums for a function with respect to a given partition and, thus, approximating the integral. Recall the each upper sum is an upper bound on the integral and each lower sum is a lower bound. Second, it illustrates the general fact that if one refines a partition—in other words, if one adds ticks to a partition—then the upper sum and lower sum can only improve. Here, “improve” means that the upper sum can only get smaller and the lower sum can only get larger. Thus, the upper sum and lower sum can only get closer to each other, providing a better approximations of the integral.

Each problem below refers the function

$$f(x) = -\frac{1}{4}x^2 + 3$$

on the interval $[-2, 2]$, as pictured below.



Graph of $f(x) = -\frac{1}{4}x^2 + 3$.

We are interested in approximating the area under f on this interval. The two partitions we will consider are

$$P = \{-2, -1, 0, 2\} \quad \text{and} \quad Q = \{-2, 1, 0, 1, 2\}.$$

Problem 1. The subintervals of P are $[-2, -1]$, $[-1, 0]$, and $[0, 2]$. (Note that the their lengths are 1, 1, and 2, respectively. So the bases for the rectangles considered for the upper and lower sums are not all equal.)

Upper sum:

(a) Compute the least upper bounds for f on each subinterval:

$$M_1 = \operatorname{lub} f([-2,-1]), \quad M_2 = \operatorname{lub} f([-1,0]), \quad M_3 = \operatorname{lub} f([0,2]).$$

Recall that, for instance, $f([-2,-1])$ is the set of numbers obtained by evaluating f at every number in $[-2,-1]$,

$$f([-2,-1]) := \{f(x) : -2 \leq x \leq -1\}.$$

It's the set of all "heights" of points on that portion of the graph of f sitting above the interval $[-2,-1]$. (Warning: the least upper bounds don't always occur at the right endpoints, and they don't always occur at the left endpoints. The same comment will apply the greatest lower bounds, later.)

(b) Compute the upper sum $U(f,P)$ for f for this partition, an overestimate of the area.

(c) Draw a picture of the graph of f and of the three rectangles whose areas appear as summands in $U(f,P)$.

Problem 2. Lower sum:

(a) Compute m_1, m_2, and m_3 the greatest lower bounds for f on each subinterval:

$$m_1 = \operatorname{glb} f([-2,-1]), \quad m_2 = \operatorname{glb} f([-1,0]), \quad m_3 = \operatorname{glb} f([0,2]).$$

(b) Compute the lower sum $L(f,P)$ for f for this partition, an underestimate of the area.

(c) Draw a picture of the graph of f and of the rectangles whose areas appear as summands in $L(f,P)$.

Problem 3. Now consider the partition $Q = \{-2,-1,0,1,2\}$.

(a) What are the subintervals for this partition? (In this case, each subinterval has length 1.)

(b) Using the notation from class, compute the M_i and m_i for these subintervals (the least upper bounds and greatest lower bounds for f on each subinterval).

(c) Compute the upper and lower sums $U(f, Q)$ and $L(f, Q)$.

(d) Draw two pictures: one of the graph of f and of the rectangles whose areas appear as summands in $U(f, Q)$, and another that does the same for $L(f, Q)$.

Problem 4. Order the set of five numbers

$$\left\{\frac{32}{3}, U(f, P), L(f, P), U(f, Q), L(f, Q)\right\}$$

from least to greatest. (The number $32/3$ is the actual area under the graph. There is a general principle that says if you refine a partition by adding tick marks, the upper sums and lower sums get better, i.e., they get closer to the actual integral. This means the upper sums should get smaller and the lower sums should get larger when you refine a partition. You should see that here.)

Problem 5. Find a function $g(x)$ whose derivative is $f(x) = -x^2/4 + 3$, and compute $g(2) - g(-2)$, the net change in g over the interval $[-2, 2]$.