Math 111 Homework for Friday, Week 3

PROBLEM 1. For this problem (and only this problem), you do not have to explain your reasoning. Just provide solutions based on your intuition. Evaluate the following limits.

(a) $\lim_{x\to -3}(3x+2)$.

(b) $\displaystyle\lim_{x\to 3}\frac{2x-3}{x+5}$.

(c) $\displaystyle\lim_{x\to 1}\sin\left(\frac{\pi x}{2}\right)$.

(d) Suppose $f(x)$ and $g(x)$ are functions and that we know $\lim_{x\to c} f(x)=\frac{3}{2}$ and $\lim_{x\to c} g(x)=\frac{1}{2}$. Evaluate the following:

(i) $\lim_{x\to c} 4\,f(x)$.

(ii) $\lim_{x\to c}(2f(x)+g(x)+4f(x)g(x))$.

(iii) $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)}$.

In class, we presented the following results:

Limit Theorem. Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

(LTa). $\lim_{x\to c}(f(x)+g(x))=\lim_{x\to c} f(x)+\lim_{x\to c} g(x)$.

(LTb). $\lim_{x\to c} f(x)g(x)=\lim_{x\to c} f(x)\,\lim_{x\to c} g(x)$.

(LTc). If $\lim_{x\to c} g(x)\neq 0$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Proposition 1. If $\lim_{x\to c} f(x)$ exists and $k\in\mathbb{R}$, then

$$\lim_{x\to c} kf(x)=k\lim_{x\to c} f(x).$$

Proposition 2. Let c and k be real numbers. Then

(a) $\lim_{x\to c} k = k$.

(b) $\lim_{x\to c} x = c$.

PROBLEM 2. Fill in each of the blanks in the argument below with one of LTa, LTb, LTc, Prop. 1, or Prop. 2 in order to justify the step in the calculation.

$$
\begin{aligned}
\lim_{x\to 2} \frac{x^2+4}{x^2-3x} &= \frac{\lim_{x\to 2}(x^2+4)}{\lim_{x\to 2}(x^2-3x)} && (\underline{\hspace{3cm}}) \\
&= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} 4}{\lim_{x\to 2} x^2 + \lim_{x\to 2}(-3x)} && (\underline{\hspace{3cm}}) \\
&= \frac{\lim_{x\to 2} x \lim_{x\to 2} x + \lim_{x\to 2} 4}{\lim_{x\to 2} x \lim_{x\to 2} x + \lim_{x\to 2}(-3x)} && (\underline{\hspace{3cm}}) \\
&= \frac{\lim_{x\to 2} x \lim_{x\to 2} x + \lim_{x\to 2} 4}{\lim_{x\to 2} x \lim_{x\to 2} x - 3\lim_{x\to 2} x} && (\underline{\hspace{3cm}}) \\
&= \frac{2\cdot 2+4}{2\cdot 2-3\cdot 2} && (\underline{\hspace{3cm}}) \\
&= -4.
\end{aligned}
$$

Here is a model solution (using the cancellation trick) which you are asked to emulate in the next problem.

Compute $\lim_{x\to 1} \dfrac{x^3 - x}{x - 1}$.

Solution. Using our limit theorems, we have

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3 - x}{x - 1} &= \lim_{x\to 1} \frac{x(x^2 - 1)}{x - 1} \\
&= \lim_{x\to 1} \frac{x(x-1)(x+1)}{x - 1} \\
&= \lim_{x\to 1} x(x+1) \\
&= \lim_{x\to 1} x \lim_{x\to 1} (x+1) \\
&= 1 \cdot 2 = 2.
\end{aligned}$$

Problem 3. Compute the following limits using the cancellation trick, the rationalization trick, and our limit theorems. Try to make your solution mimic the style presented in the model solution, above, as closely as possible. (**Note:** Be careful not to drop $\lim_{x\to c}$ across equal signs until the last step.)

(a) $\lim_{x\to 0} \dfrac{x}{x^2 - x}$.

(b) $\lim_{x\to 2} \dfrac{2x^2 - x - 3}{x + 1}$.

(c) $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$. (You will need to recall how to factor a difference of cubes.)

(d) $\lim_{x\to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$.

(e) $\lim_{x\to 16} \dfrac{4 - \sqrt{x}}{x - 16}$.

Problem 4. Suppose that $\lim_{x\to 0} f(x) = 1$. Use the definition of the limit with $\varepsilon = 1$ to show that there must be some open interval about 0 such that $f(x) > 0$ for every x in that interval, except possibly at $x = 0$.

[Hints: Start this problem by looking up our definition of the limit. What does it say in our situation? Recall that saying $0 < |x - c| < \delta$, is the same as saying x is in the interval $(c - \delta, c + \delta)$ and $x \neq c$, and saying $|f(x) - L| < \varepsilon$ is the same thing as saying $f(x)$ is in the interval $(L - \varepsilon, L + \varepsilon)$. Most important: take a look at the upper picture on page 2 of my lecture notes for Friday, Week 1. Drawing that picture in the case where $\lim_{x\to 0} f(x) = 1$ should point the way to the solution.]