Math 111 Homework for Friday, Week 1

Please read the following and then turn in the problems at the end.

I. Lines. The *slope* of the line passing between points (x_1, y_1) and (x_2, y_2) in the plane is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

(The equality shows that the slope does not depend on the order in which we list the two points—either formula serves as a definition for the slope.) The *standard form* for the equation of a *non-vertical* line in the plane has the form $y = mx + b$ for some real numbers m and b. In that case, the line is the set of points (x, y) such that $y = mx + b$. In other words, the set of points $(x, mx + b)$ where x ranges over the real numbers. The proposition below will show that in this case, the number m is the slope of the line. The standard form for the equation of a *vertical* line is $x = c$ for some real number c. It describes the set of points (c, y) as y ranges over the real numbers. For a vertical line, we will say that the slope is undefined or that the slope is infinite.

Proposition. Suppose that the points (x_1, y_1) and (x_2, y_2) lie on the (non-vertical) line with standard equation $y = mx + b$. Then m is the slope of the line and the line meets the y-axis at the point $(0, b)$.

Proof. Since both points lie on the line, they both satisfy the line's equation:

$$y_1 = mx_1 + b$$
$$y_2 = mx_2 + b.$$

Subtracting the first equation from the second, we get

$$y_2 - y_1 = (mx_2 + b) - (mx_1 + b) = mx_2 - mx_1 = m(x_2 - x_1).$$

Since the line is not vertical, we know that $x_2 - x_1 \neq 0$. So we can solve for m to find that m is the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Next, every point on the y-axis has the form $(0, a)$ for some real number a. If this point is to lie on the line, it must satisfy the line's equation:

$$a = m \cdot 0 + b = b.$$

Therefore, the point at which the line meets the y-axis is $(0, b)$, as claimed (since we just showed that $a = b$). $\square$

Practice and further reading: See our textbook, Section 1.2.1.

II. Absolute value and distance. The absolute value of a real number x, denoted $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For example, according to the above definition $|-3| = -(-3) = 3$ since $-3 < 0$.

The *distance* between two points a and b on the real number line [1] is $|a - b|$. The next proposition says that the distance from a to b is the same as the distance from b to a.

Proposition. Let a and b be real numbers. Then $|a - b| = |b - a|$.

Proof. If $a = b$, the $|a - b| = |b - a| = 0$, and the result holds. So suppose that $a \neq b$. Without loss of generality, we may assume that $a > b$. It then follows that $a - b > 0$, and according to the definition of absolute value, $|a - b| = a - b$. On the other hand, $a > b$ also implies that $b - a < 0$. So from the definition of absolute value, we have $|b - a| = -(b - a) = a - b$.

We have just seen that, in any case, $|a - b| = |b - a|$. $\square$

A useful result, which we will state here without proof, is that if x and a are real numbers, then

$$|x| \leq a \quad \text{if and only if} \quad -a \leq x \leq a.$$

Similarly,

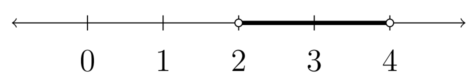
$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

[1] We call a and b *points* instead of *numbers* because we are now thinking geometrically.

Note that if $a < 0$, then there are no x satisfying the inequalities. To illustrate the use of the above, say you want to describe the set of points x such that $|x - 3| < 1$. Then we can compute:

$$\begin{aligned} |x-3| < 1 &\Leftrightarrow -1 < x - 3 < 1 \\ &\Leftrightarrow 3 - 1 < 3 + (x-3) < 3 + 1 \quad \text{(adding 3 across the inequalities)} \\ &\Leftrightarrow 2 < x < 4. \end{aligned}$$

(The symbol $\Leftrightarrow$ is shorthand for "if and only if".) Thus, the points x satisfying $|x - 3| < 1$ are exactly the points strictly between 2 and 4. Note that the solution coincides with our geometric intuition: according to our definition of distance, the points x satisfying $|x - 3| < 1$ are exactly the points whose distance from 3 is less than 1. Our intuition tells us that these are exactly the points between 2 and 4:



Practice and further reading: link.

Problems to turn in for homework are on the next page.

Please turn in the following problems using Gradescope. See our Moodle page and our class information sheet for instructions on using Gradescope. Remember the following guidelines:

- Do not turn in scratch work! After finding a solution, copy it neatly onto the paper you will scan for Gradescope.
- Turn in a pdf to Gradescope, not a jpeg or other image file.
- For Gradescope to work properly, each problem needs to be matched with a page in the work you turn in. Otherwise, it is possible the grader will not see it.
- Show your work. Neatly display the steps you make to find your solutions. If a verbal argument is called for, use *complete sentences*.
- Make sure your scan is easy to read—not too dark or blurry.

A final pointer: after performing a calculation, it is sometimes easy to check that your solution is correct (for instance, in Problem 1, below).

Problems

PROBLEM 1. Find equations in standard form, either $y = mx + b$ or $x = c$, for the following lines. (To find the equation, you need to find m and b, or find c.)

(a) The line through the points $(2, 4)$ and $(-3, 1)$.

(b) The line through the points $(4, 2)$ and $(4, 5)$.

(c) The line with slope 3 and containing the point $(1, -4)$.

(d) The line through $(1, 2)$ and $(1 + h, 2 + 2h + h^2)$ where $h \neq 0$. (Find formulas for m and b, as functions of h, simplified as much as possible.)

PROBLEM 2. For each of the following, (i) find a and b such that $a < x < b$ and draw this set of points x on the real number line.

(a) $|x - 5| < 3$.

(b) $|x + 4| < 1$.

PROBLEM 3. We defined the distance between two points a and b to be $|a-b|$. How would you describe the set of points x satisfying $|x + 4| < 1$ in terms of distances?