Math 111

November 30, 2022

Today

Differential equations:

- Warm-up
- Population models I

Warm-up

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Solution:

$$y(t) = ae^{kt}$$

where $a = y(0)$ is the initial population size.

Exponential growth model

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Answer:

$$t = \frac{\ln(2)}{k}.$$

Inversely, proportional to k.

Population model based on Newton's law of cooling

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where r and S are positive constants.

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- How could this be a model for heating/cooling?

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&\Rightarrow \quad \int \frac{y'(t)}{S - y(t)}\, dt = rt + c
\end{aligned}$$

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$$\int \frac{y'(t)}{S - y(t)}\, dt = -\int \frac{du}{u} = -\ln(u) = -\ln(S - y(t)).$$

Equation $(\star)$ becomes:

$$-\ln(S - y(t)) = rt + c.$$

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Exponentiate both sides of the equation:

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$$y(t) = S - \frac{1}{a}e^{-rt}.$$

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Sanity check: what happens as $t \to \infty$?

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Let's write the solution

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Does this solution make sense?

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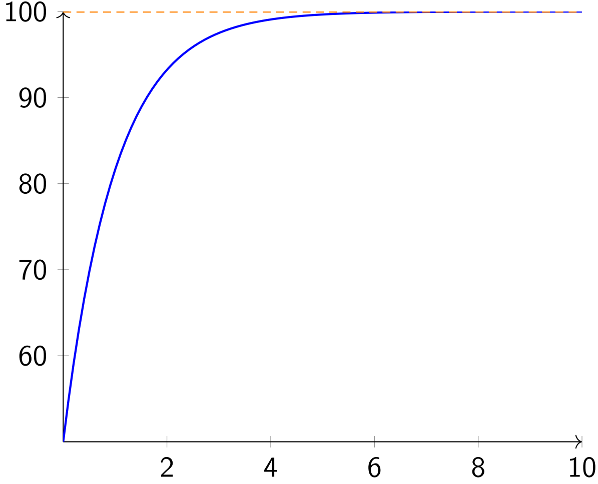
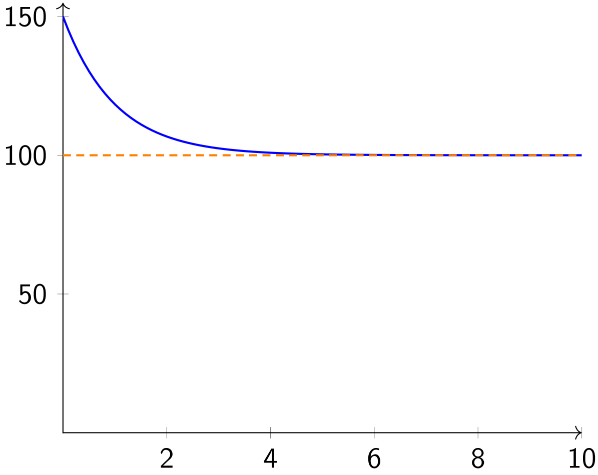


Graph of $y(t) = S + (I - S)e^{-rt}$ with $S = 100$, $I = 50$, and $r = 1$.

Population model based on Newton's law of cooling



Graph of $y(t) = S + (I - S)e^{-rt}$ with $S = 100$, $I = 150$, and $r = 1$.