Math 111

November 28, 2022

Fundamental theorem of calculus

For those missing last Wednesday's class:

Carefully read the lecture notes for that day. Start by reviewing the definition of the integral and the statement of the mean value theorem in the Addendum.

You should be able to understand *every* step.

Today

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Differential equations:

- Examples
- Initial conditions

First example

Let $y = y(t)$ be a function of t and consider the *second-order* differential equation

$$y''(t) - y(t) = 0.$$

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Check solution on board.

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What do expect the behavior of the particle to be for various initial conditions?

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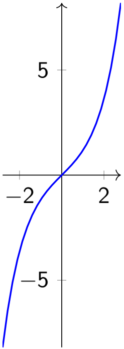

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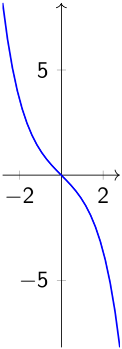

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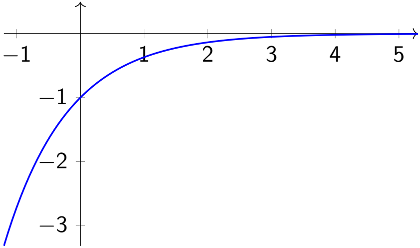

Graph of $y(t) = -e^{-t}$.

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$$y(t) = 0.$$

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Integrate:

$$\int \frac{y'}{y}\,dt = \int \frac{3}{t}\,dt.$$

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For instance, suppose $y(1) = 1$. Then

$$y(t) = t^3.$$