Math 111

December 2, 2022

Today

- Logistic model for population growth.

Logistic growth model

$P(t) =$ size of a population at time t

The logistic growth model is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

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The logistic growth model is

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right).$$

- When is the population increasing? decreasing?
- What if $P(t)$ is very small?

Solution.

Goal: solve the logistic growth differential equation for P.

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The equation is separable:

$$P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right) \quad \Rightarrow \quad \frac{P'(t)}{P(t)\left(1 - \frac{P(t)}{K}\right)} = r.$$

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We need a new technique now. It's called the method of *partial fractions*.

The idea is to find constants A and B such that

$$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1 - \frac{P(t)}{K}}.$$

Solution

$$\frac{1}{P(t)\left(1-\frac{P(t)}{K}\right)} = \frac{A}{P(t)} + \frac{B}{1-\frac{P(t)}{K}}$$

Solution

$$\frac{1}{P(t)\left(1-\frac{P(t)}{K}\right)}=\frac{A}{P(t)}+\frac{B}{1-\frac{P(t)}{K}}$$

Get a common denominator for the RHS:

$$\frac{A}{P(t)}+\frac{B(t)}{1-\frac{P(t)}{K}}=\frac{A\left(1-\frac{P(t)}{K}\right)+BP(t)}{P(t)\left(1-\frac{P(t)}{K}\right)}$$

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Therefore,

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we must have

$$1 = A\left(1-\frac{P(t)}{K}\right)+BP(t) = A-\frac{A}{K}P(t)+BP(t).$$

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Gather terms involving P:

$$1 = A+\left(-\frac{A}{K}+B\right)P(t).$$

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$$A = 1 \quad \text{and} \quad -\frac{A}{K}+B = 0.$$

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So $A = 1$ and $B = \frac{1}{K}$.

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Now compute the LHS using our partial fraction decomposition.

Solution

We have found that

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Therefore,

$$\int \frac{P'(t)}{P(t)\left(1-\frac{P(t)}{K}\right)}\,dt = \int \frac{P'(t)}{P(t)} + \frac{P'(t)/K}{1-\frac{P(t)}{K}}\,dt$$

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Solution

We are left with the integral

$$\frac{1}{K} \int \frac{P'(t)}{1 - \frac{P(t)}{K}} \, dt$$

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$$\frac{1}{K}\int \frac{P'(t)}{1-\frac{P(t)}{K}}\,dt$$

Let $u = 1 - P(t)/K$. Then $du = -\frac{1}{K}P'(t)\,dt$.

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We are left with the integral

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$$= -\ln(u) + \text{constant}$$

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$$\begin{aligned}
\frac{1}{K}\int \frac{P'(t)}{1-\frac{P(t)}{K}}\,dt &= -\int \frac{du}{u} \\
&= -\ln(u) + \text{constant} \\
&= -\ln\left(1-\frac{P(t)}{K}\right) + \text{constant}.
\end{aligned}$$

Solution

Putting this all together

$$\ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) = \ln P(t) + \ln\left(\left(1 - \frac{P(t)}{K}\right)^{-1}\right)$$

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Putting this all together

$$\begin{aligned}\ln P(t) - \ln\left(1 - \frac{P(t)}{K}\right) &= \ln P(t) + \ln\left(\left(1 - \frac{P(t)}{K}\right)^{-1}\right) \\ &= \ln\left(P(t)\left(1 - \frac{P(t)}{K}\right)^{-1}\right)\end{aligned}$$

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Exponentiate both sides to get

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Exponentiate both sides to get

$$P(t)\left(1 - \frac{P(t)}{K}\right)^{-1} = e^{rt}e^{\mathrm{constant}} = ae^{rt}$$

for some positive constant a.

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$$\Rightarrow P(t)=\frac{aKe^{rt}}{ae^{rt}+K}\Rightarrow P(t)=\frac{aK}{a+Ke^{-rt}}.$$

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$$a = \frac{P(0)K}{K - P(0)}.$$

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Substituting this expression for a and simplifying gives the final form for the solution

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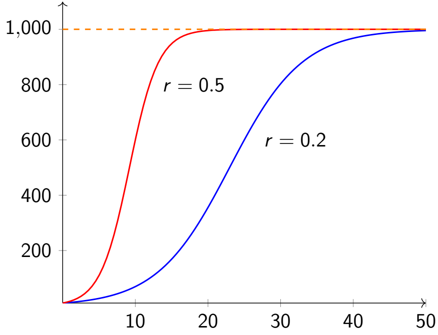

Graph of $P(t)$ with $K = 1000$ and $P(0) = 10$ and two different growth rates: $r = 0.5$ in red and $r = 0.2$ in blue.

More on partial fractions

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Example. Compute the indefinite integral

$$\int \frac{x}{x^2 - 1} \, dx.$$

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The derivative of the denominator is $2x$, which is in the numerator (up to a constant).

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The derivative of the denominator is $2x$, which is in the numerator (up to a constant).

This integral can be handled with the substitution $u = x^2 - 1$.

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$x^2 - 1 = (x + 1)(x - 1)$. So try partial fractions.

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Example. Compute the indefinite integral

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This time the derivative of the denominator does not appear in the numerator. Does the denominator factor? Yes: $x^2-1=(x+1)(x-1)$. So try partial fractions.

$$\frac{1}{x^2-1}=\frac{1}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}.$$

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The problem is to find the constants A and B.

More on partial fractions

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$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)},$$

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then comparing numerators on both sides, we must have, for all x,
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Set $x = 1$ to get rid of the second term:

$$1 = A(1+1) + B(1-1) = 2A$$

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Now set $x = -1$ to get rid of the first term:

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Now set $x = -1$ to get rid of the first term:

$$1 = A(-1+1) + B(-1-1) = -2B \quad \Rightarrow \quad B = -\frac{1}{2}.$$

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Finally,

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{-1/2}{x-1}\,dx$$

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$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x+1} + \frac{-1/2}{x-1}\,dx \\ &= \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x-1}\,dx\end{aligned}$$

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&= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) + c \\
&= \frac{1}{2}(\ln(x+1) - \ln(x-1)) + c
\end{aligned}
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&= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) + c \\
&= \frac{1}{2}(\ln(x+1) - \ln(x-1)) + c \\
&= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + c.
\end{aligned}$$