# Math 111

November 23, 2022

# Today

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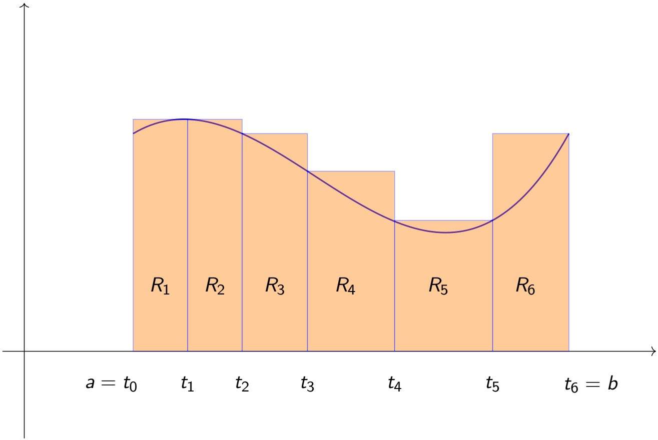
Proof of the Fundamental Theorem of Calculus.

# Definition of the integral




An upper sum $U(f, P)$ for some function $f$.

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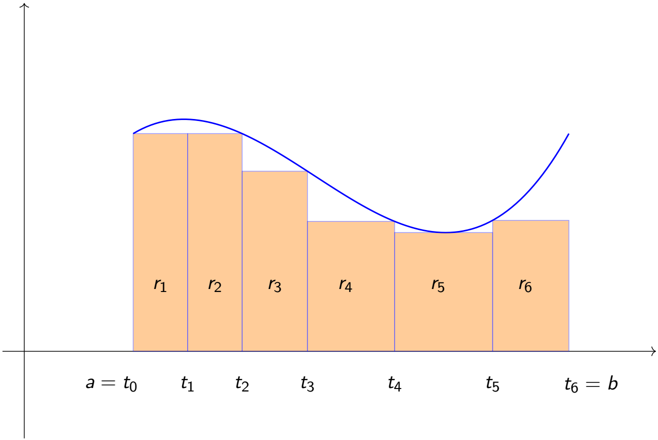


A lower sum $L(f, P)$ for some function $f$.

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Partition of $[a, b]$: $P = \{t_0, t_1, \ldots, t_n\}$

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Definition: $f$ is *integrable* if $U\int f = L\int f$, in which case

$$\int_a^b f := U\int f = L\int f.$$

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If $g' = f$, then

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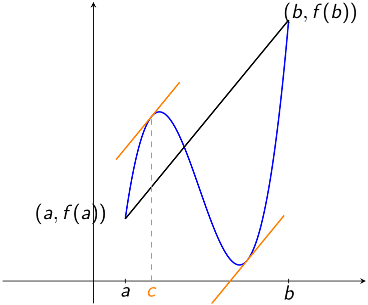
Apply the Mean Value Theorem to $g(x)$ on each subinterval $[t_{i-1}, t_i]$:

# Mean value theorem

**Mean Value Theorem (MVT).** Let $f$ be a continuous function on $[a, b]$ and differentiable on $(a, b)$. Then there exists a number $c$ with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



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Apply the Mean Value Theorem to $g(x)$ on each subinterval $[t_{i-1}, t_i]$. For each $i = 1, \ldots, n$, we get $c_i \in [t_{i-1}, t_i]$ such that

$$g'(c_i) = \frac{g(t_i) - g(t_{i-1})}{t_i - t_{i-1}}.$$

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Why is it true that for each $i$,

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Note to self: record this last equation on the blackboard.

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So

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$$L\int_a^b f \leq g(b) - g(a) \leq U\int_a^b f.$$

Since $f$ is integrable,

$$L\int_a^b f = U\int_a^b f = \int_a^b f(x)\,dx.$$

Therefore,

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

□