Math 111

November 21, 2022

Today

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- The mean value theorem.
- Question about inverses from last time.

Mean value theorem

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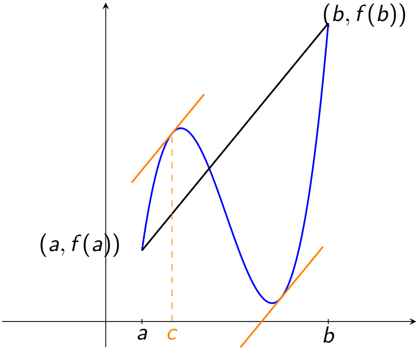
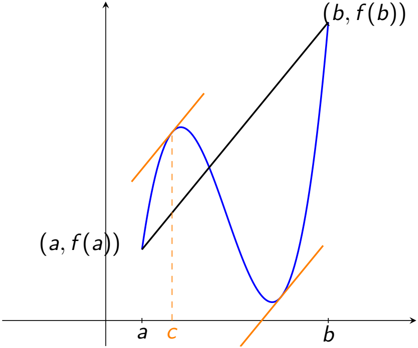
What would the graph of distance versus time look like?

Are we certain that your speed was exactly 100 mph at some point?

Mean value theorem



Mean value theorem



Find an equation expressing what you see about slopes of lines:

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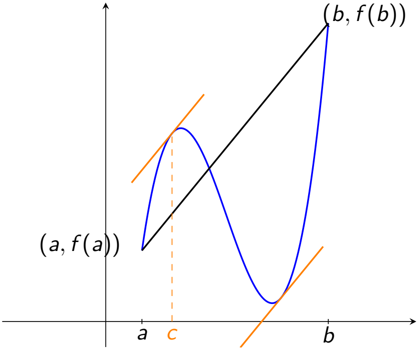

Find an equation expressing what you see about slopes of lines:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Mean value theorem

Mean Value Theorem (MVT). Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists a number c with $a < c < b$ such that

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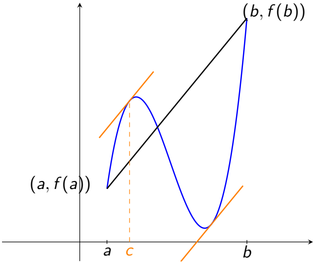

Mean value theorem (with two possible choices for c).

Proof. Math 112. □

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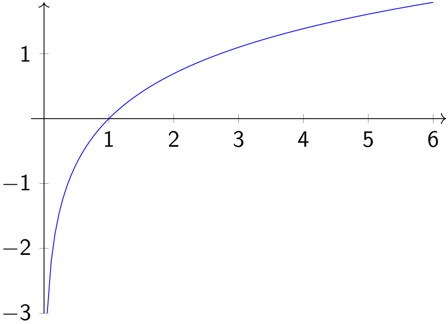
Proofs on the board (and in our notes).

Review of ln and exp from last week

For $x > 0$,

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$



Graph of $\ln(x)$.

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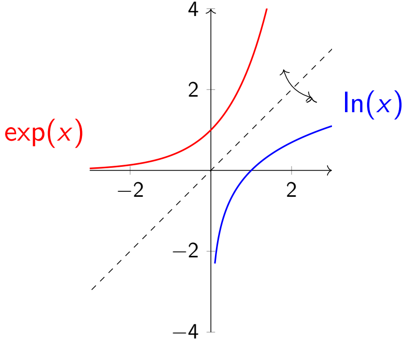

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The exponential converts sums to products:

$$\exp(x + y) = \exp(x)\exp(y).$$

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Example. $2^\pi = e^{\pi \ln(2)} = \exp(\pi \ln(2)) \approx 8.82.$

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Recall from last time:

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