Math 111

November 16, 2022

Today

Today

- Properties of the natural logarithm.
- Differentiation and integration examples using the logarithm.

FTC

Recall our two versions of the fundamental theorem of calculus:

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FTC Version II: If

$$g(x) = \int_a^x f(t)\,dt$$

for each $x \in I$, then $g'(x) = f(x)$ for each $x \in I$.

Definition

Recall the definition of the natural logarithm from last time:

Definition. For $x > 0$, the *natural logarithm* is

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Property 1.

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Proof. This follows directly from FTC2:

$$(\ln(x))' = \left(\int_1^x \frac{1}{t}\,dt\right)' \overset{\text{FTC2}}{=} \frac{1}{x}.$$

$\square$

Properties of $\ln(x)$

Property 2. $\ln(x)$ is an increasing function (i.e., it has positive slope) and its graph is concave down.

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Increasing: $\ln'(x) = \frac{1}{x} > 0$.

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Property 2. $\ln(x)$ is an increasing function (i.e., it has positive slope) and its graph is concave down.

Proof.

Increasing: $\ln'(x) = \frac{1}{x} > 0$.

Concave down: $\ln''(x) = -\frac{1}{x^2} < 0$.

Extension of the definition of the integral

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Example.

$$\int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

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Property 3.

$$\ln(x) \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

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Proof. These properties all follow directly from the definition

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

and the fact that $\frac{1}{t} > 0$ for $t > 0$.

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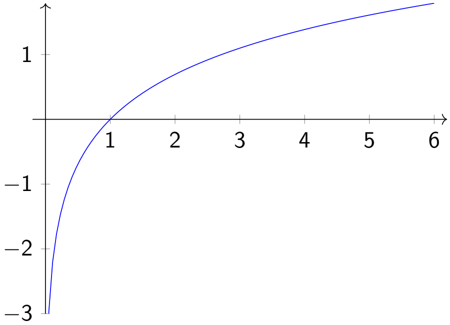
and the fact that $\frac{1}{t} > 0$ for $t > 0$. For instance, when $0 < x < 1$,

$$\ln(x) = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0.$$

Properties of $\ln(x)$

Property 4. The graph of $\ln(x)$:



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So $\ln(1) = \ln(1) + \ln(1)$.

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So $\ln(1) = \ln(1) + \ln(1)$. Subtracting $\ln(1)$ from both sides shows that $\ln(1) = 0$, consistent with what we already know.

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Proof.

$$
\begin{aligned}
\ln(x^0) &= \ln(1) = 0 = 0 \cdot \ln(x) \\
\ln(x^1) &= \ln(x) = 1 \cdot \ln(x) \\
\ln(x^2) &= \ln(x \cdot x) = \ln(x) + \ln(x) = 2\ln(x) \\
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\ln(x^4) &= \ln(x \cdot x^3) = \ln(x) + \ln(x^3) = \ln(x) + 3\ln(x) = 4\ln(x),
\end{aligned}
$$

and so on. Formally, one would prove the complete result using induction. $\square$

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Proof. Let n be a positive integer. We have $x^n \cdot x^{-n} = 1$. Therefore,

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Proof. Let n be a positive integer. We have $x^n \cdot x^{-n} = 1$. Therefore,

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So $n\ln(x) + \ln(x^{-n}) = 0$, and the result follows. $\square$

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Proof. We have proved the result for α any integer. For arbitrary real numbers, see Math 112. The problem is knowing the definition of x^α when α is an arbitrary real number. $\square$

Properties of $\ln(x)$

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Proof. Homework. $\square$

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We know that $\ln(2) > 0$.

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See our lecture notes for solutions.