Math 111

November 14, 2022

Today

Today

- Short quiz.
- Integration by parts example.
- Second version of fundamental theorem of calculus.
- Definition of the natural logarithm.

Quiz

- Let f be a function defined in an open interval about real number c. What is the definition of the derivative, $f'(c)$?
- Suppose $f(x) = 3x^2 + 5$. Use the definition of the derivative to compute $f'(2)$.

Quiz solution

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Solution. The derivative is defined by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

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Solution. We have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Quiz solution

Suppose $f(x) = 3x^2 + 5$. Use the definition of the derivative to compute $f'(2)$.

Solution. We have

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(3(2+h)^2 + 5) - (3\cdot 2^2 + 5)}{h} \end{aligned}$$

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&= \lim_{h \to 0} \frac{(3(2+h)^2 + 5) - (3 \cdot 2^2 + 5)}{h} \\
&= \lim_{h \to 0} \frac{(3(4 + 4h + h^2) + 5) - (3 \cdot 4 + 5)}{h}
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&= \lim_{h\to 0} \frac{(3\cdot 4 + 12h + 3h^2 + 5) - (3\cdot 4 + 5)}{h}
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&= \lim_{h \to 0} \frac{(3 \cdot 4 + 12h + 3h^2 + 5) - (3 \cdot 4 + 5)}{h} \\
&= \lim_{h \to 0} \frac{12h + 3h^2}{h} \\
&= \lim_{h \to 0} 12 + 3h = 12.
\end{aligned}$$

Integration by parts example

Recall the formula for integration by parts:

$$\int u \, dv = uv - \int v \, du.$$

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To proceed we need to compute

$$\int e^x \sin(x)\,dx.$$

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It looks like we've gone around in circles, but checking carefully...

Integration by parts example

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$$\begin{aligned}\int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - \left(-e^x \cos(x) + \int e^x \cos(x)\,dx\right)\end{aligned}$$

Integration by parts example

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So

$$\int e^x \cos(x)\,dx = \frac{1}{2}(\sin(x) + \cos(x))e^x + c.$$

Two versions of the fundamental theorem of calculus

FTC Version I: $\int_a^b g'(x)\,dx = g(b) - g(a)$

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Fine point: What does the integral mean when $x < a$?

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Example.

$$\int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

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$$\int_0^3 x^2\,dx = \frac{x^3}{3}\bigg|_0^3 = 9 \quad \text{and} \quad \int_3^0 x^2\,dx = -\int_0^3 x^2\,dx = -9.$$

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$$g(x) = \int_a^x f(t)\,dt \quad \Rightarrow \quad g'(x) = f(x).$$

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Example. Let $f(x) = x^5$, and let $a = 0$.

Then define g by

$$\begin{aligned} g(x) &= \int_0^x t^5\,dt \\ &= \frac{1}{6}t^6\Big|_{t=0}^{x} \end{aligned}$$

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Check:

$$g'(x) = \left(\frac{1}{6}x^6\right)' = x^5 = f(x).$$

Example of FTC II

Let

$$b(x) = \int_3^x e^{-t^2+\cos(t)}\,dt.$$

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The natural logarithm

Consider these functions and their derivatives:

$f(x)$	x^{-3}	x^{-2}	x^{-1}	1	x	x^2	x^3
$f'(x)$	$-3x^{-4}$	$-2x^{-3}$	$-x^{-2}$	0	1	$2x$	$3x^2$

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So, forgetting $+c$,

$$\int \frac{1}{x^4}\,dx = -\frac{1}{3}\cdot\frac{1}{x^3},$$

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$$\int 0\,dx = 1,$$

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$$\int 0\,dx = 1, \quad \int x\,dx = \frac{1}{2}x^2, \quad \int x^2\,dx = \frac{1}{3}x^3.$$

Question: What about $\int \frac{1}{x}\,dx$?

The natural logarithm

Definition. For $x > 0$, the *natural logarithm* is

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

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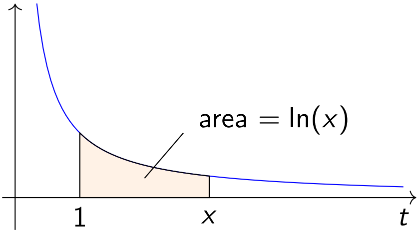
By FTC II, it follows that

$$\ln'(x) = \frac{1}{x}.$$

The natural logarithm

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$



Graph of $f(t) = \frac{1}{t}$.