Math 111

November 18, 2022

Today

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- The inverse function theorem (IFT).
- The exponential function.

The inverse function theorem

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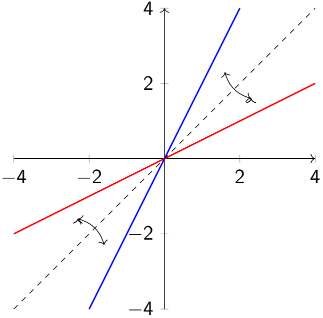
Examples.



Graphs of inverse functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$.

The inverse function theorem

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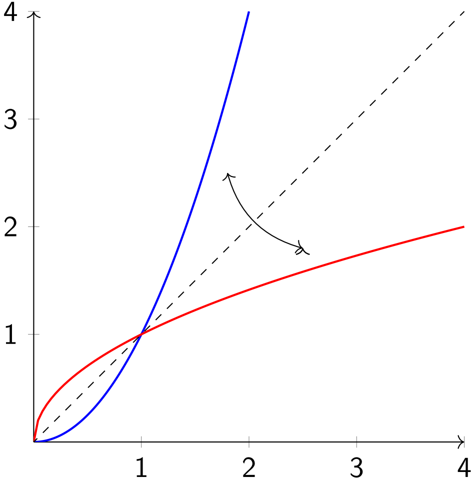

Graphs of inverse functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

One-to-one functions

Definition. A function f is *one-to-one* if $x \neq y$ implies $f(x) \neq f(y)$.

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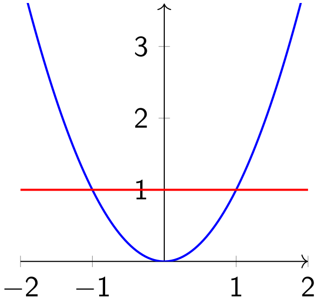
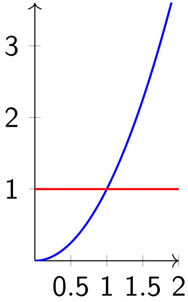
Horizontal line test:



$f(x) = x^2$ fails the horizontal line test on $(-\infty, \infty)$.

One-to-one functions

If we restrict $f(x) = x^2$ to be a function on $[0, \infty)$, it is one-to-one:



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Proposition. If the function f is one-to-one, it has an inverse.

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Example. Considering $f(x) = x^2$ as a function on $[0, \infty)$, then it has an inverse: $g(x) = \sqrt{x}$.

The inverse function theorem

Theorem. (Inverse function theorem, (IFT).) Suppose f is differentiable, and suppose f has an inverse g. Then g is differentiable and

$$g'(x) = \frac{1}{f'(g(x))}$$

provided $f'(g(x)) \neq 0$.

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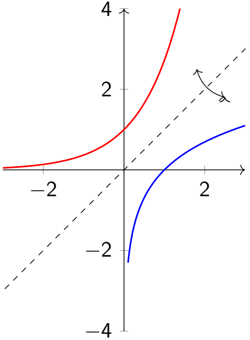

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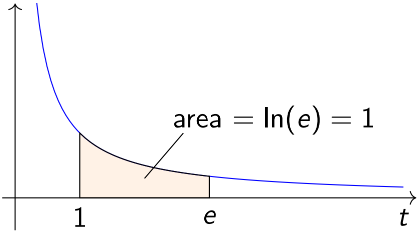

Graph of $f(t) = \frac{1}{t}$.

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And so on:

$$\exp(n) = e^n$$

for $n = 0, 1, 2, \dots$

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Example. $2^\pi = e^{\pi \ln(2)} \approx 8.82.$

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- $\ln(a^x) = x \ln(a)$.