Math 111

November 9, 2022

Today

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- Antiderivatives and the fundamental theorem of calculus (FTC).
- First properties of the integral.

Antiderivatives

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The antiderivatives of f are exactly functions of the form

$$2x^3 + c$$

where $c \in \mathbb{R}$.

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\int_0^2 6x^2 &= \int_0^2 (2x^3 + 17)' \, dx \\
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&= \Big(2(2)^3 + 17\Big) - \Big(2(0)^3 + 17\Big) \\
&= 2(2)^3 - 2(0)^3 \\
&= 16.
\end{aligned}$$

Properties of the integral

Suppose that f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$. then

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b cf = c\int_a^b f.$$

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$$\int_0^2 2x^2 - 7\cos(x)\,dx = 2\int_0^2 x^2\,dx - 7\int_0^2 \cos(x)\,dx.$$

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Suppose that f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $\int_a^b f \leq \int_a^b g$. In other words, integration preserves inequalities.

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Example. We have $5\cos(x) \leq 5$ on $[3, 8]$. Therefore,

$$\int_3^8 \cos(x)\, dx \leq \int_3^8 5\, dx.$$

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Example.

$$\int_0^{10} e^{3x}\, dx = \int_0^4 e^{3x}\, dx + \int_4^{10} e^{3x}\, dx.$$

Properties of the integral

If f is continuous on $[a, b]$, then $\int_a^b f$ exists. In general,

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Example. The function $f(x) = |x|$ is continuous but is not differentiable at $x = 0$. The function

$$g(x) = \begin{cases} x & \text{if } x \neq 0, \\ 5 & \text{if } x = 0. \end{cases}$$

us integrable but not continuous.

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3. $\int_0^{2\pi} \sin(x)\,dx$.
4. $\int_0^2 (3x^2 + 2x + 1)(x^3 + x^2 + x)^{99}\,dx$.

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8. $\int x(3x^2 + 5)^{100} \, dx$.
9. $\int x^2 e^{x^3} \, dx$.