Math 111

November 7, 2022

Today

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- Fundamental theorem of calculus (FTC)

Review definition of the integral

From homework:

$$f(x) = -\frac{1}{4}x^2 + 3$$

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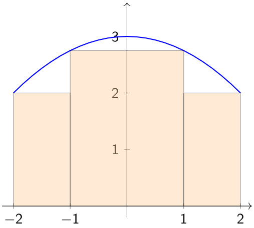
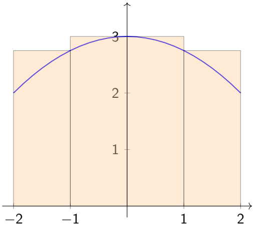


$L(f, Q)$

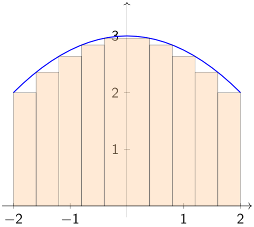


$U(f, Q)$

$$L(f, Q) = 9.5 < 11.5 = U(f, Q).$$

Review of the integral

$$R = \{-2, -1.6, -1.2, -0.8, -0.4, 0.4, 0.8, 1.2, 1.6, 2\}$$



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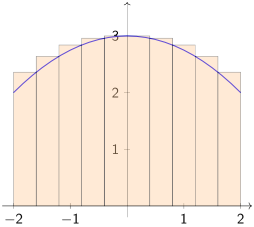

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$$L(f, R) = 10.24 < 11.04 = U(f, R).$$

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4. Then
$$A = \lim_{n\to\infty} L(f, P_n) \leq L\int f \leq U\int f \leq \lim_{n\to\infty} U(f, P_n) = A.$$
5. So $\int_a^b f = A$.

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Notation making variable explicit:

$$\int_a^b f(x)\, dx = g(b) - g(a).$$

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$$\begin{aligned}\int_{-2}^{2} f &= g(2) - g(-2) \\ &= \left(-\frac{1}{12}(2)^3 + 3(2)\right) - \left(-\frac{1}{12}(-2)^3 + 3(-2)\right)\end{aligned}$$

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&= \frac{32}{3} = 10.666\ldots
\end{aligned}$$

Some notation

$$\int_{-2}^{2} -\frac{1}{4}x^2 + 3\, dx$$

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5. $f(x) = \cos(x) + \sin(x)$ on $[0, \pi/2]$.

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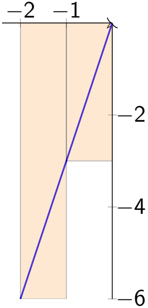
$$= -\frac{3}{2} \cdot 4$$

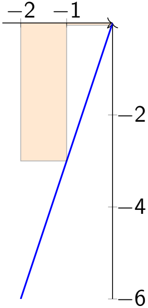
$$= -6.$$

How can the area be negative?

Signed area

Lower and upper sums for the partition $P = \{-2, -1, 0\}$:





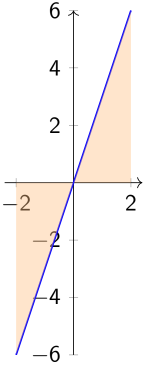
$L(f, P) = -6 - 3 = -9$ $\qquad$ $U(f, P) = -3 - 0 = -3.$

Signed area

The integral calculates area between the graph and the x-axis positively when the graph is above the x-axis and negatively when the graph is below the x-axis.

Example

Graph of $f(x) = 3x$ on $[-2, 2]$:



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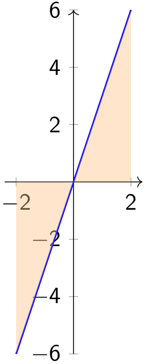

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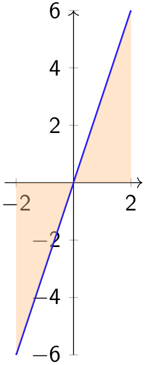

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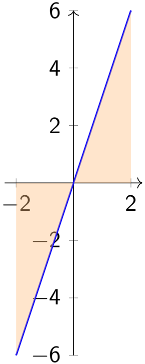

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