Math 111

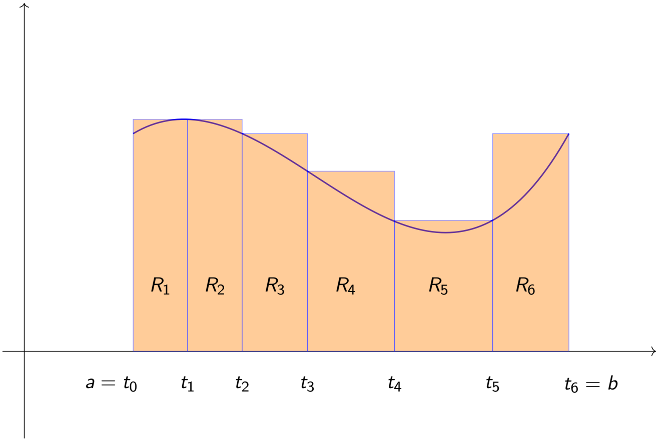
November 2, 2022

Today

Today

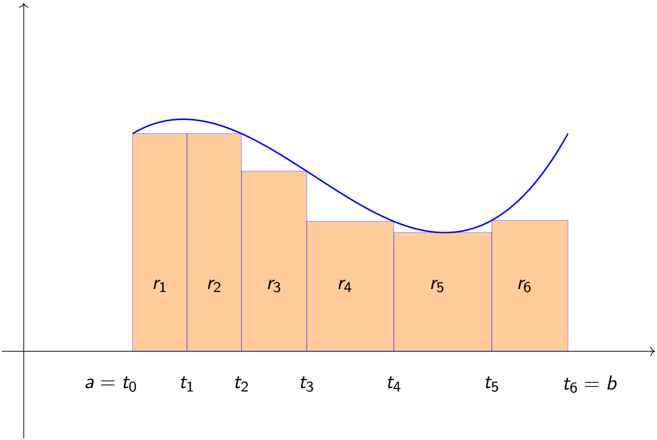
- Review definition of the integral.
- Examples.

Upper sum



An upper sum $U(f, P)$ for some function f.

Lower sum



A lower sum $L(f, P)$ for some function f.

Definition of the integral

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Define the upper integral to be the greatest lower bound of this set:

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Define the lower integral to be the least upper bound of this set:

$$L\int_a^b f = \mathrm{lub}\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

Definition of the integral

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If the lower and upper integrals are equal, we define the integral of f on $[a, b]$ to be their common value:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Definition of the integral

- Partition of a close interval $[a, b]$:

$$P = \{t_0, \ldots, t_n\}$$

with

$$a = t_0 < t_1 < \cdots < t_n = b.$$

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- The subintervals of the partition P:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n].$$

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- The y-values for f on the i-th interval:

$$f([t_{i-1}, t_i]).$$

This is the set of heights of the graph of the function sitting over the interval $[t_{i-1}, t_i]$. .

Definition of the integral

- $$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$

  These are the heights for the best over-estimating rectangle and under-estimating rectangle, respectively.

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  $$\begin{aligned} L(f, P) &= m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) \\ &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \end{aligned}$$

  These are over- and under-estimates for the integral.

Definition of an integral

- Upper and lower integrals:

$$U\int_a^b f := \mathrm{glb}\,\{U(f,P) : P \text{ a partition of } [a,b]\}$$

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$$L\int_a^b f := \operatorname{lub}\{L(f,P) : P \text{ a partition of } [a,b]\}.$$

- If $U\int_a^b f = L\int_a^b f$, the f is integrable and

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Example

$$f(x) = \begin{cases} -(x-1)^2 + 1 & \text{if } x < \frac{3}{2}, \\ x - \frac{3}{4} & \text{if } x \geq \frac{3}{2}. \end{cases}$$

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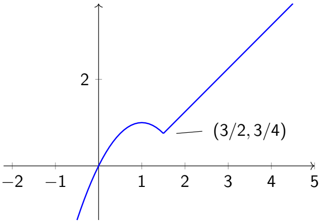

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- Form the upper and lower sums:

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L(f, P) &= m_1(t_1 - t_0) + m_2(t_2 - t_1) + \cdots + m_n(t_n - t_{n-1}) \\
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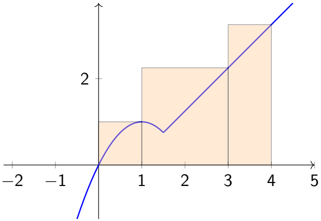

$U(f,P)$

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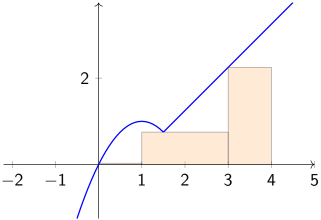

$L(f, P)$

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The actual area under f from $x = 0$ to $x = 4$ is

$$\int_0^4 f = \frac{49}{8} = 6.125.$$

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We see

$$L(f, P) \leq \int_0^4 f \leq U(f, P)$$

since

$$3.75 \leq 6.125 \leq 8.75.$$

Examples

Percentage error of upper and lower sums compared to true value of the integral:

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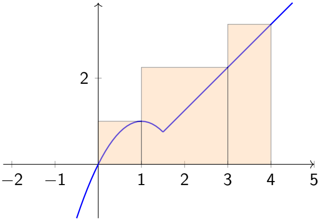
Percentage error of upper and lower sums compared to true value of the integral:

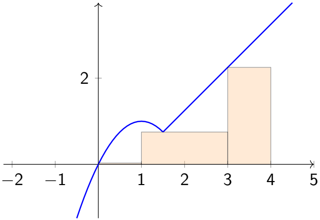
$$\text{upper sum:} \quad \frac{8.75 - 6.125}{6.125} \approx 42.9\%$$

$$\text{lower sum:} \quad \frac{6.125 - 3.75}{6.125} \approx 38.6\%.$$

Examples



$U(f, P)$



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