Math 111

October 31, 2022

Today

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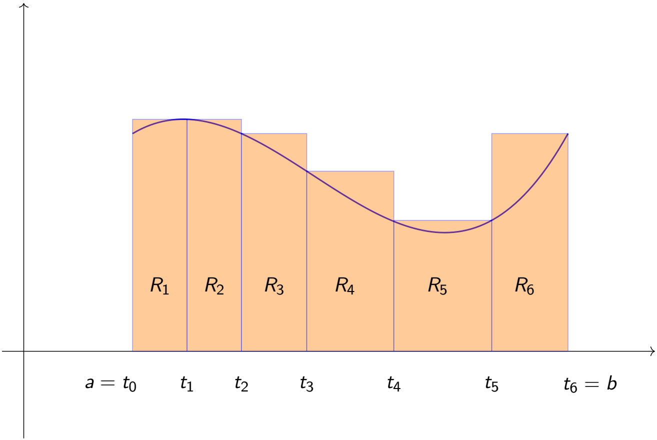
- Definition of the integral.

Upper sum



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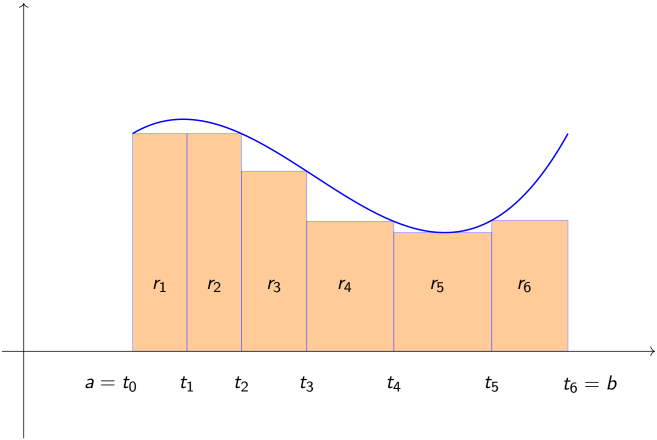
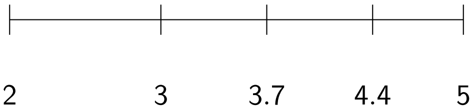
Lower sum



A lower sum $L(f, P)$ for some function f.

Partitions

A partition of the interval $[2, 5]$:



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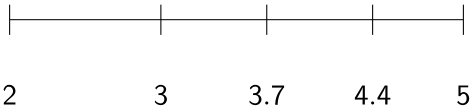

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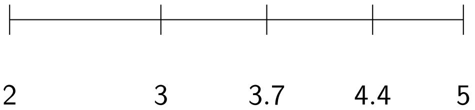

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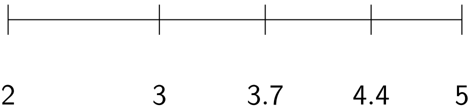

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Subintervals of the partition P:

$$[2, 3], \quad [3, 3.7], \quad [3.7, 4.4], \quad [4.4, 5]$$

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Consider partition of $[0, 1]$ given by

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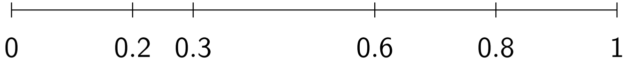

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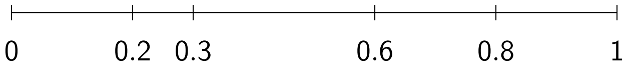

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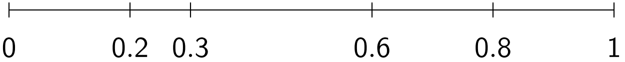

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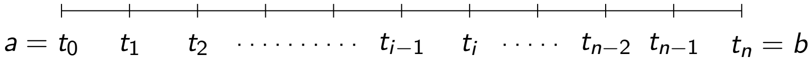

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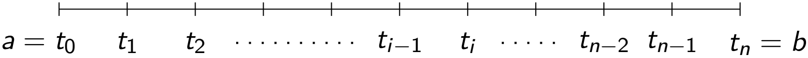

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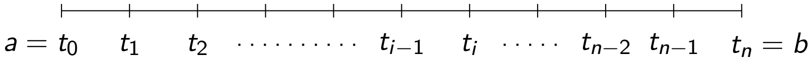

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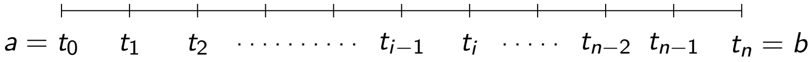

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The *i*-th subinterval: $[t_{i-1}, t_i]$.

The image of an interval

The image of an interval $[s, t]$ under a function f:

$$f([s, t]) = \{f(x) : s \leq x \leq t\}.$$

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Thus, $f([s, t])$ is the set of all heights $f(x)$ for x in the interval $[s, t]$.

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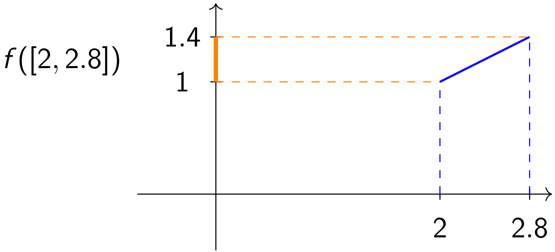

The image of the interval $[2, 2.8]$ under $f(x) = \dfrac{x}{2}$.

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Example. Say $[s, t] = [2, 2.8]$ and $f(x) = \frac{x}{2}$

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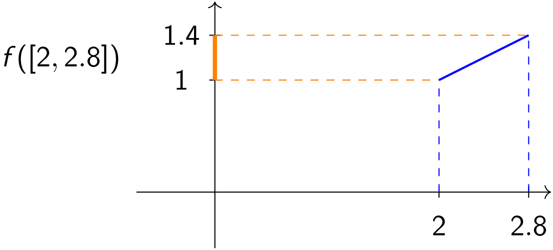

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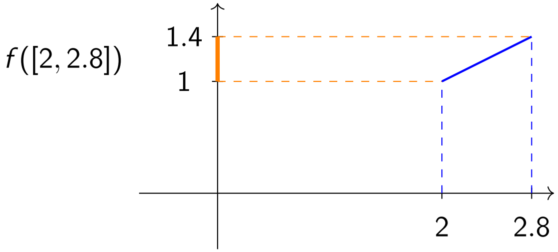

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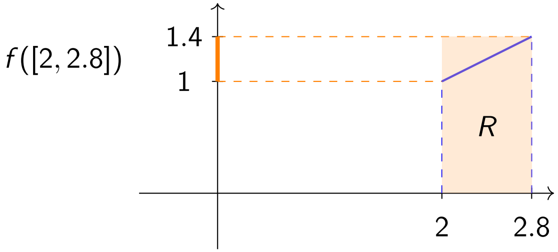

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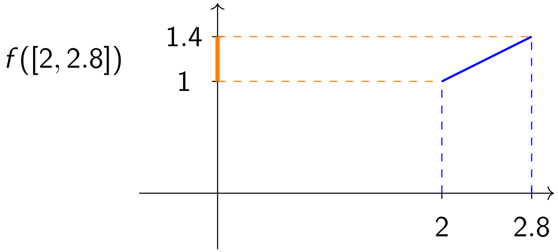

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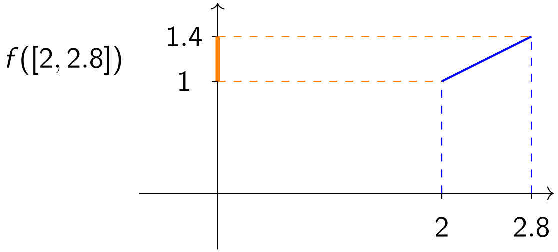

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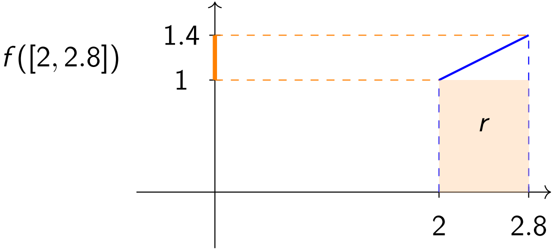

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That way, these heights will always exist.

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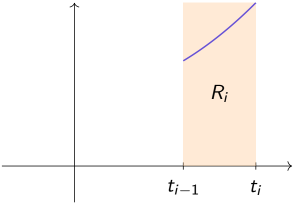

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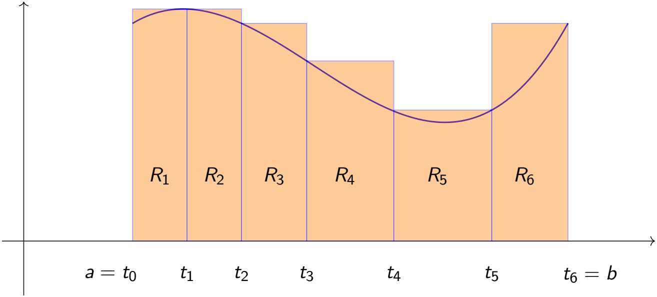

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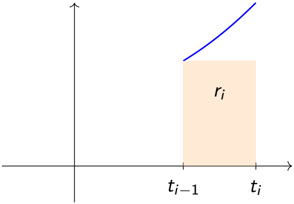

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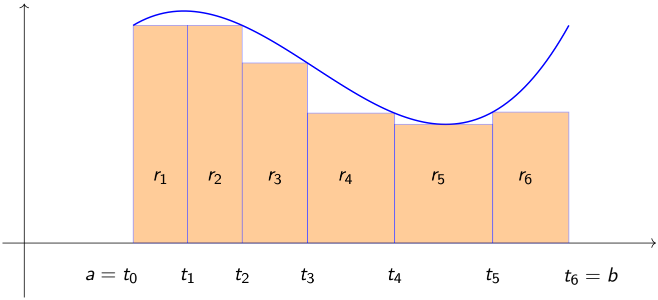

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Define the upper integral to be the greatest lower bound of this set:

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Define the lower integral to be the least upper bound of this set:

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If the lower and upper integrals are equal, we define the integral of f on $[a, b]$ to be their common value:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Definition of the integral

Review the definition in the lecture notes for today.