Math 111

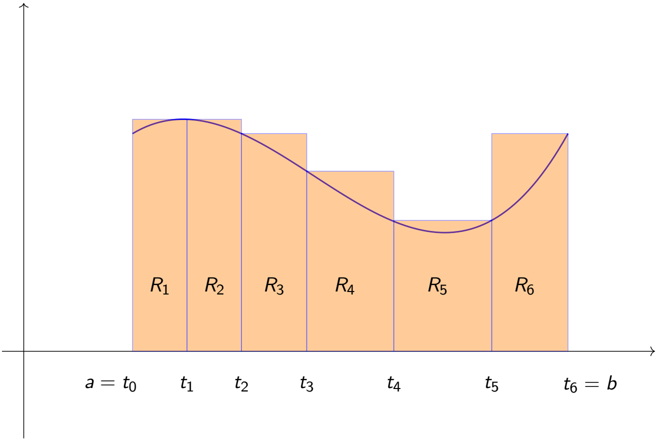
November 4, 2022

Today

Today

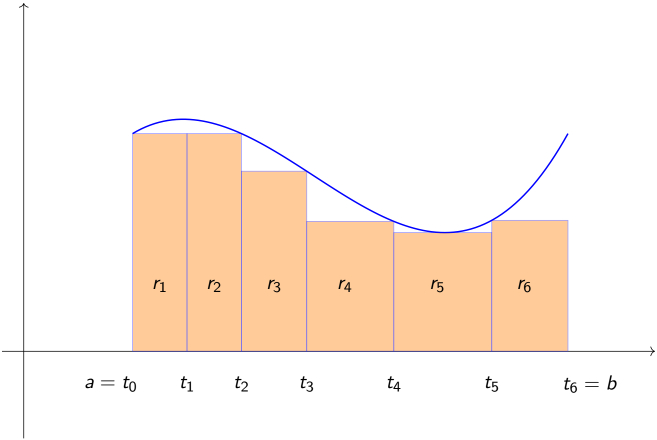
- Examples involving the definition of the integral.

Upper sum



An upper sum $U(f, P)$ for some function f.

Lower sum



A lower sum $L(f, P)$ for some function f.

Definition of the integral

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$$\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

Define the upper integral to be the greatest lower bound of this set:

$$U\int_a^b f = \mathrm{glb}\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

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Each lower sum is a number. Collect these numbers in a set:

$$\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Define the lower integral to be the least upper bound of this set:

$$L\int_a^b f = \mathrm{lub}\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Definition of the integral

We always have

$$L\int_a^b f \leq U\int_a^b f.$$

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If the lower and upper integrals are equal, we define the integral of f on $[a, b]$ to be their common value:

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Definition of the integral

- Partition of a close interval $[a, b]$:

$$P = \{t_0, \ldots, t_n\}$$

with

$$a = t_0 < t_1 < \cdots < t_n = b.$$

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- The subintervals of the partition P:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n].$$

  The i-th subinterval is $[t_{i-1}, t_i]$. It's length is $t_i - t_{i-1}$. You should think of each of these as a base for a rectangle.

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- The y-values for f on the i-th interval:

$$f([t_{i-1}, t_i]).$$

  This is the set of heights of the graph of the function sitting over the interval $[t_{i-1}, t_i]$. .

Definition of the integral

- $$M_i = \operatorname{lub} f([t_{i-1}, t_i]) \quad \text{and} \quad m_i = \operatorname{glb} f([t_{i-1}, t_i]).$$

  These are the heights for the best over-estimating rectangle and under-estimating rectangle, respectively.

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  These are over- and under-estimates for the integral.

Definition of an integral

- Upper and lower integrals:

$$U\int_a^b f := \mathrm{glb}\,\{U(f,P) : P \text{ a partition of } [a,b]\}$$

$$L\int_a^b f := \mathrm{lub}\,\{L(f,P) : P \text{ a partition of } [a,b]\}\,.$$

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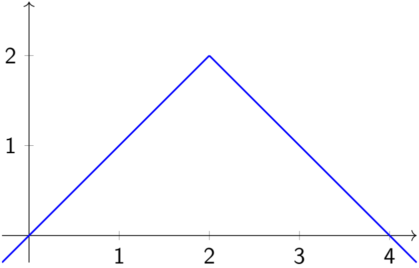
- If $U\int_a^b f = L\int_a^b f$, the f is integrable and

$$\int_a^b f := L\int_a^b f = U\int_a^b f.$$

Example I



Graph of $f(x) = 2 - |x - 2|$.

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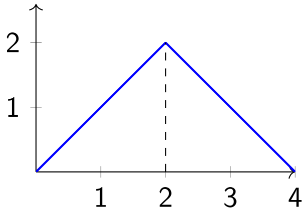

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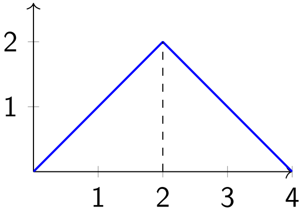
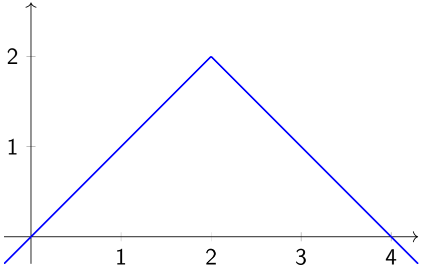


$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \cdot 4 \cdot 2 = 4.$$

Example I

Compute the upper and lower sums for f for the partition

$$P = \{0, 1, 3, 4\}.$$



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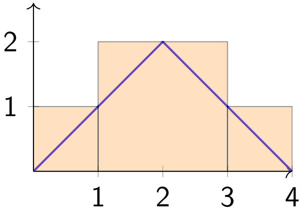

Overestimate $U(f, P)$.

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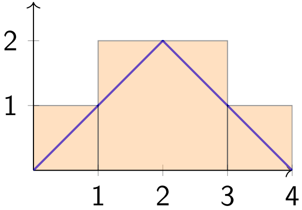
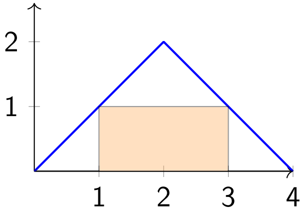


Overestimate $U(f, P)$.

$$
\begin{aligned}
U(f, P) &= M_1(1-0) + M_2(3-1) + M_3(4-3) \\
&= 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \\
&= 6.
\end{aligned}
$$

Example I



Underestimate $L(f, P)$.

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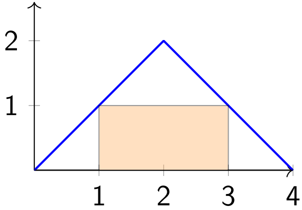


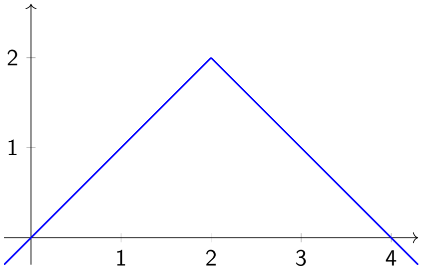
Underestimate $L(f, P)$.

$$\begin{aligned} L(f, P) &= m_1(1-0) + m_2(3-1) + m_3(4-3) \\ &= 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 \\ &= 2. \end{aligned}$$

Example I

Compute the upper and lower sums for f for the partition

$$Q = \left\{0, 1, \frac{3}{2}, 3, 4\right\}.$$



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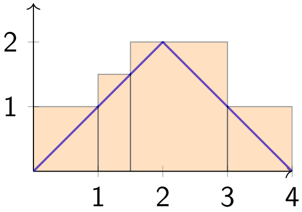

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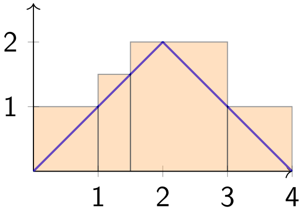

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\begin{aligned}
U(f, Q) &= M_1(1-0) + M_2(3/2-1) + M_3(3-3/2) + M_4(4-3) \\
&= 1 \cdot 1 + \frac{3}{2} \cdot \frac{1}{2} + 2 \cdot \frac{3}{2} + 1 \cdot 1 \\
&= 5\frac{3}{4} = \frac{23}{4} = 5.75.
\end{aligned}
$$

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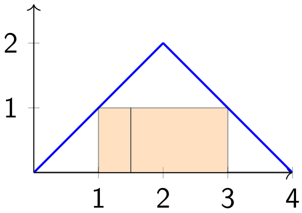

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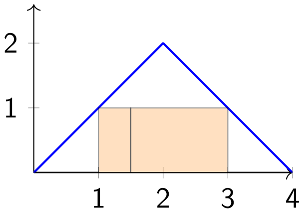

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$$L(f, Q) = m_1(1-0) + m_2(3/2-1) + m_3(3-3/2) + m_4(4-3)$$

$$= 0 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{3}{2} + 0 \cdot 1$$

$$= 2.$$

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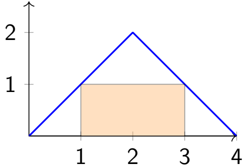

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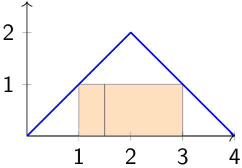

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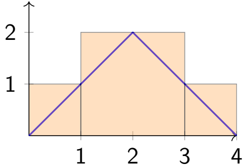

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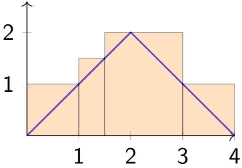


Overestimate $U(f, Q)$.

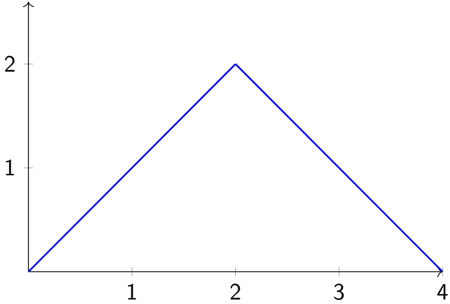
$$L(f, P) \leq L(f, Q) \leq \text{actual area } \leq U(f, Q) \leq U(f, P)$$

is

$$2 \leq 2 \leq 4 \leq 5.75 \leq 6.$$

Example I

Which partition gives the worst upper estimate? worst lower estimates?



Graph of $f(x) = 2 - |x - 2|$.

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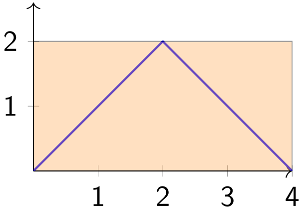

Overestimate $U(f, S)$.

$$U(f, S) = M_1(4 - 0) = 2 \cdot 4 = 8.$$

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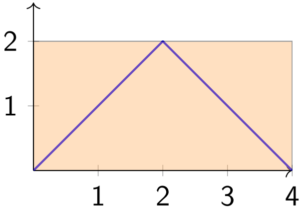

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$$U(f, S) = M_1(4 - 0) = 2 \cdot 4 = 8.$$

$$L(f, S) = m_1(4 - 0) = 0 \cdot 4 = 0.$$

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Describe the set of real numbers consisting of all possible upper sums for f. (What is the worst estimate? What is the best estimate?)

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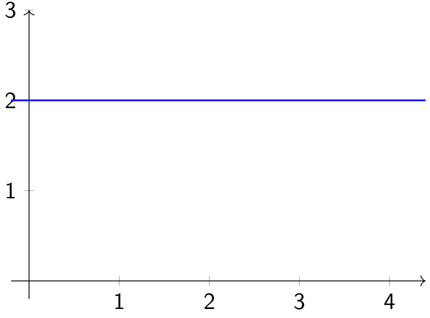
$$L\int_0^4 f = \mathrm{lub}[0, 4) = 4 = U\int_0^4 f = \mathrm{glb}(4, 8].$$

Example II

Consider the function the constant function $f(x) = 2$. Here is a graph:



Graph of $f(x) = 2$.

Use the definition of the integral to prove that $\int_1^4 f = 6$.

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By definition of the integral, this means

$$\int_1^4 f = 6.$$

Example III

$$\begin{aligned} f\colon [0,1] &\to [0,1] \\ x &\mapsto \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases} \end{aligned}$$

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$$f\colon [0,1] \to [0,1]$$
$$x \mapsto \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

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$$

So the integral does not exist.