Math 111

October 24, 2022

Today

Today

- Start midterm exam review.
- Curve sketching.

Midterm

1. Definition of limit. [51%]
2. ε- δ proof. [54%]
3. Definition of continuity. [74%]
4. Definition of the derivative. [71%]
5. Compute derivative using the definition. [55%]
6. Calculate some derivatives (product rule, quotient rule, chain rule, etc.) [78%]
7. Calculate equation of a tangent line. [80%]
8. Proof of sum rule for limits. [41%]

Definition of a limit

From the Midterm review sheet:

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- ⋆ You will be asked the definition of the limit (Friday, Week 1). You should practice the definition by writing it from memory on a sheet of paper and comparing with the actual definition until you get it perfectly. Changing almost any part of the definition will break it!

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From the exam:

Note: In the following "what does it mean to say" is code for "what is the definition of". It does not mean "what is your intuitive understanding of".

Definition of the limit

Problem 1. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. What does it mean to say that $\lim_{x \to c} f(x) = L$.

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Solution. It means that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if x satisfies $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Definition of continuity

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Continuity.

- ⋆ You will be asked the definition of continuity (Wednesday, Week 3).

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Definition of continuity

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Problem 3. What does it mean to say that a function $f(x)$ is continuous at a point c?

Solution. It means that $\lim_{x \to c} f(x) = f(c)$.

Quiz at the beginning of the next class period

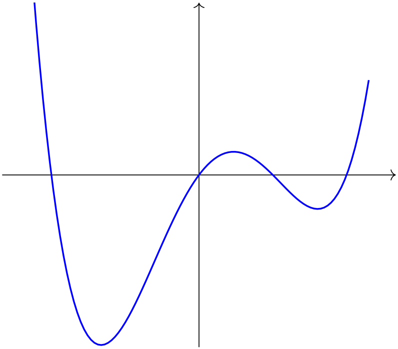
We will have a quiz at the beginning of the next class (Wednesday) consisting of Problem 1 and Problem 3 from the midterm.

Reminder

Technology. The use of electronic devices (computers, cell phones, tablets, etc.) is not allowed in the classroom without my authorization. Browsing the internet, answering your email, and texting during class is rude—it disrupts learning. It distracts your classmates and your instructor. Talk to me if you have a specific reason for needing to use technology (for example, note-taking).

Warm-up

For the function pictured below, what are the signs of $f'(x)$ and $f''(x)$ at each point x?



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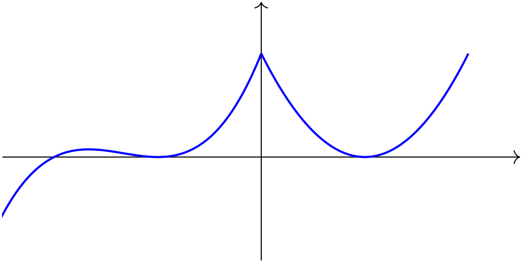

Checklist for curve tracing using derivatives

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1. Calculate where $f'(x) = 0$ and where $f'(x)$ does not exist (i.e., find the *critical points* of f), and evaluate the function at these points. Draw these points in your graph.

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4. What happens to $f(x)$ as $x \to \infty$ and as $x \to -\infty$? (Are there horizontal asymptotes?)
5. Find places where the function "blows up", i.e., find any vertical asymptotes. If you find a vertical asymptote, how does f behave on either side of it?

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5. Find places where the function "blows up", i.e., find any vertical asymptotes. If you find a vertical asymptote, how does f behave on either side of it?
6. To determine concavity, you can use the second derivative: $f''(x) > 0 \Rightarrow$ concave up, and $f''(x) < 0 \Rightarrow$ concave down.

Example 1

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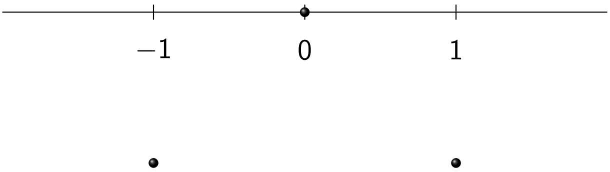
We have $f(-1) = -1$, $f(0) = 0$, and $f(1) = -1$. So we plot the points $(-1, -1)$, $(0, 0)$, and $(1, -1)$.

$f(x) = x^4 - 2x^2, \;\; f'(x) = 4x^3 - 4x.$

2. The sign of f' between critical points:



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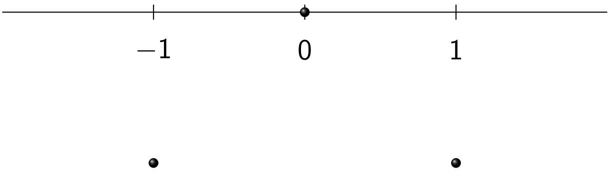
slope of f:	down	up	down	up
f':	$-$	$+$	$-$	$+$



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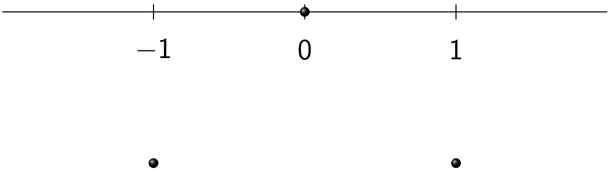

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slope of f:	down		up		down		up
f':	$-$		$+$		$-$		$+$
		-1		0		1	

2. Zeros of f:

$$f(x) = x^4 - 2x^2, \quad f'(x) = 4x^3 - 4x.$$

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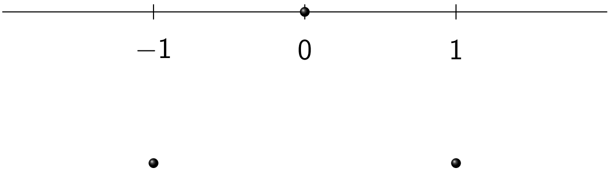

2. Zeros of f:

$$f(x) = x^4 - 4x = 0 \quad \Longleftrightarrow \quad x = -\sqrt{2}, 0, \sqrt{2}.$$

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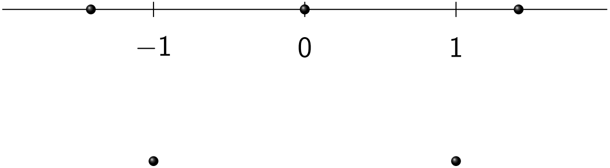

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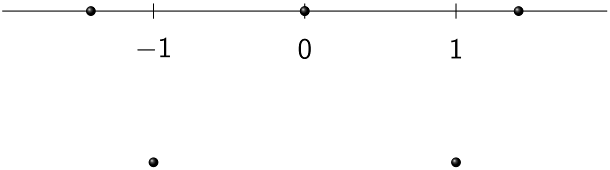

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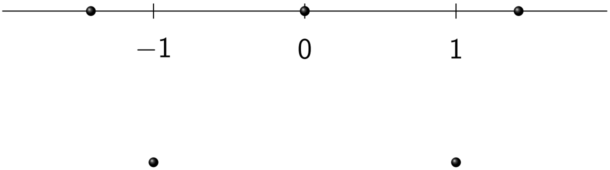

4. Horizontal asymptotes:

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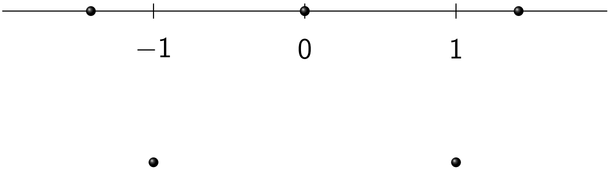

4. Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

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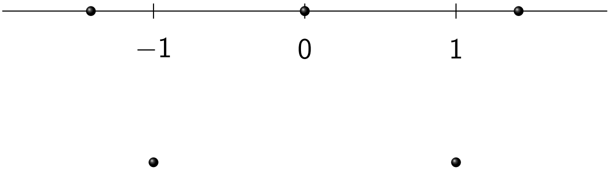

4. Horizontal asymptotes:

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty.$$

No horizontal asymptotes.

$f(x) = x^4 - 2x^2, \;\; f'(x) = 4x^3 - 4x.$

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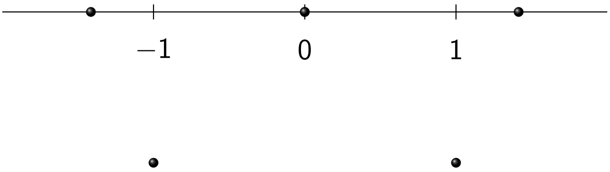

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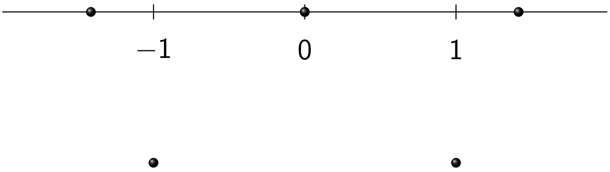

5. Vertical asymptotes (blow-up points):

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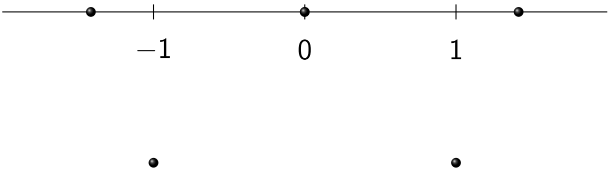

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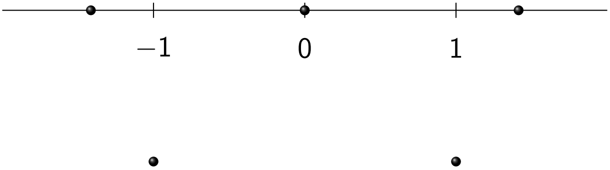

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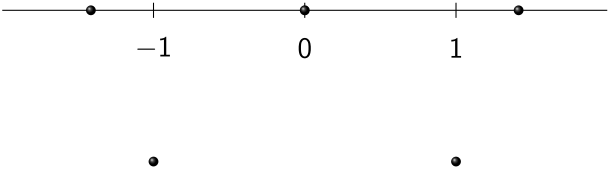

6. Concavity (where is f concave up?):

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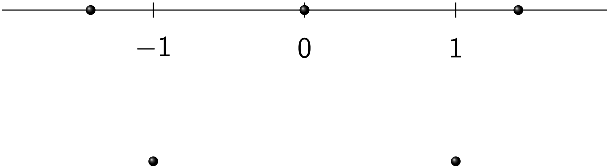

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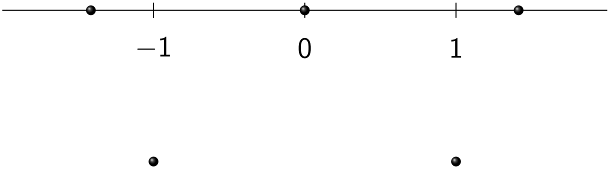

6. Concavity (where is f concave up?):

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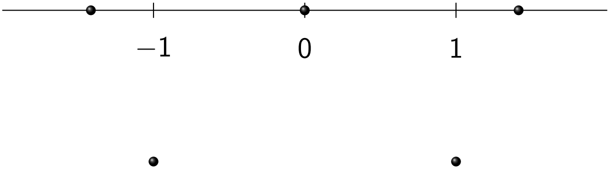

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$$f''(x) = 12x^2 - 4 > 0 \quad \Longleftrightarrow \quad x^2 > \frac{1}{3}$$

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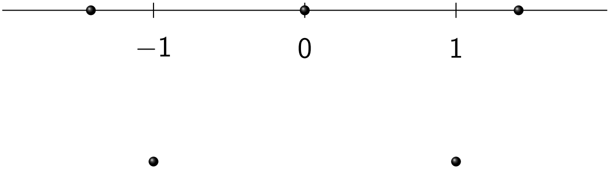

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$$\Longleftrightarrow \quad x > \sqrt{\frac{1}{3}} \quad \text{or} \quad x < -\sqrt{\frac{1}{3}}.$$

$$f(x) = x^4 - 2x^2, \;\; f'(x) = 4x^3 - 4x, \;\; f''(x) = 12x^2 - 4$$

Concavity at critical (slope $= 0$) points:

$$f(x) = x^4 - 2x^2, \;\; f'(x) = 4x^3 - 4x, \;\; f''(x) = 12x^2 - 4$$

Concavity at critical (slope $= 0$) points:

$$f''(-1) = 12(-1)^2 - 4 = 12 - 4 = 8 > 0$$

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Concavity at critical (slope $= 0$) points:

$$f''(-1) = 12(-1)^2 - 4 = 12 - 4 = 8 > 0 \quad \Rightarrow \quad \text{local minimum}$$

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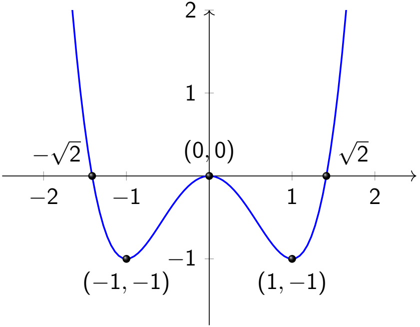
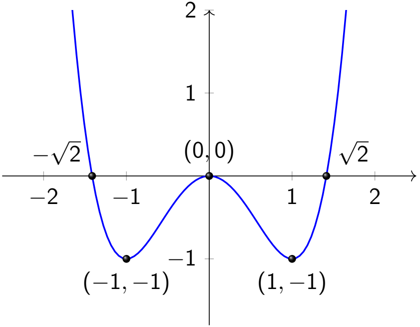
Graph of $f(x) = x^4 - 2x^2$



Graph of $f(x) = x^4 - 2x^2$

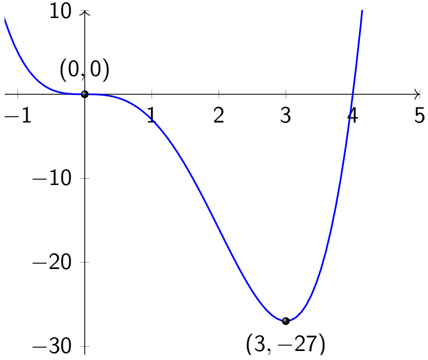


Inflection points occur where $f''(x) = 0$, i.e., at $\pm\sqrt{1/3}$.

Example 2

See our lecture notes for a similar analysis for

$$f(x) = x^4 - 4x^3 = x^3(x-4).$$



Example 3

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$$f'(x) = \frac{x(x-2)}{(x-1)^2} = 0$$

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$$f'(x) = \frac{x(x-2)}{(x-1)^2} = 0 \quad \Longleftrightarrow \quad x = 0, 2 \quad \text{and undefined at } x = 1.$$

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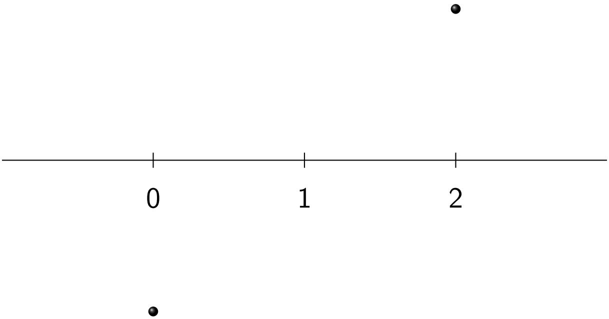
$$f'(x) = \frac{x(x-2)}{(x-1)^2} = 0 \quad \Longleftrightarrow \quad x = 0, 2 \quad \text{and undefined at } x = 1.$$

We have $f(0) = -2$, and $f(2) = 2$. So we plot the points $(0, -2)$, and $(2, 2)$.

$f(x) = \frac{x^2-2x+2}{x-1}, \;\; f'(x) = \frac{x(x-2)}{(x-1)^2}.$

2. The sign of f' between critical points:



$f'(x) = 0$ when $x = 0, 2$, undefined at $x = 1$

$f(x) = \frac{x^2-2x+2}{x-1}$, $f'(x) = \frac{x(x-2)}{(x-1)^2}$.

2. The sign of f' between critical points:

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

$f'(x) = 0$ when $x = 0, 2$, undefined at $x = 1$

$f(x) = \frac{x^2-2x+2}{x-1}$, $f'(x) = \frac{x(x-2)}{(x-1)^2}$.

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

$$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

2. Zeros of f:

$f(x) = \frac{x^2-2x+2}{x-1}$, $f'(x) = \frac{x(x-2)}{(x-1)^2}$.

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

2. Zeros of f:

$$f(x) = \frac{x^2 - 2x + 2}{x - 1} = 0 \quad \Longrightarrow \quad x^2 - 2x + 2 = 0.$$

No real zeros (use quadratic equation).

$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$

slope of f:	up	down	down	up
f':	$+$	$-$	$-$	$+$
	0	1	2	

$f(x) = \frac{x^2-2x+2}{x-1}, \;\; f'(x) = \frac{x(x-2)}{(x-1)^2}.$

slope of f:	up	down	down	up
f':	$+$	$-$	$-$	$+$
		0	1	2

4. Horizontal asymptotes:

$$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

4. Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

$f(x) = \frac{x^2-2x+2}{x-1}, \;\; f'(x) = \frac{x(x-2)}{(x-1)^2}.$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

4. Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

No horizontal asymptotes.

$$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

4. Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

No horizontal asymptotes. But note $f(x) \approx x$ for x large.

$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$

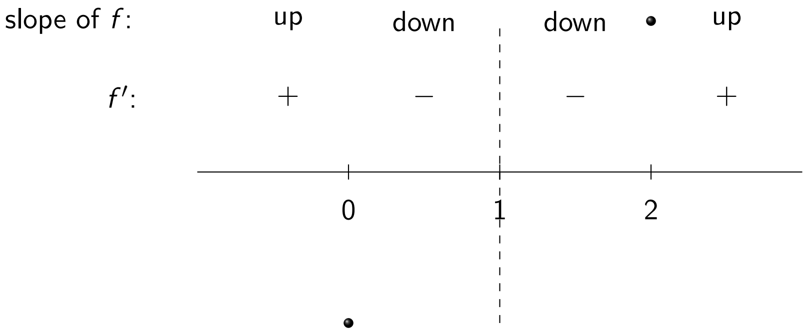
slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

5. Vertical asymptote at $x = 1$.

$$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$$

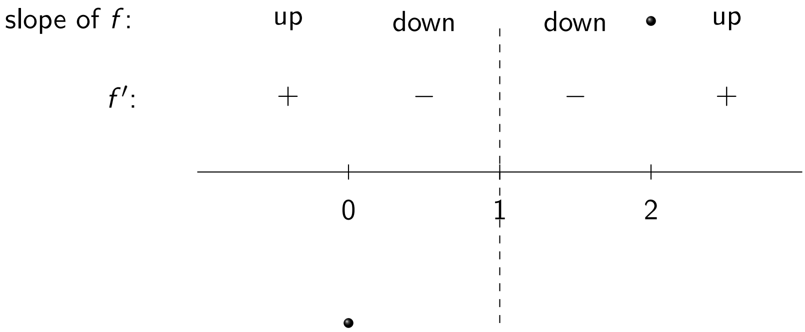


5. Vertical asymptote at $x = 1$.

$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$

slope of f:	up		down		down		up
f':	$+$		$-$		$-$		$+$
		0		1		2	

$f(x) = \frac{x^2-2x+2}{x-1}, \quad f'(x) = \frac{x(x-2)}{(x-1)^2}.$



6. Concavity skipped since it's messy.

Graph of $f(x) = \frac{x^2-2x+2}{x-1}$