Math 111

October 28, 2022

Today

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- An interesting sum.

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- A first integral.

An interesting sum.

Lemma. For each $n > 0$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

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$$1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \cdot 7}{2}.$$

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$$\begin{array}{cc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline \end{array}$$

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Idea that generalizes to give a proof of the lemma—do the sum twice:

$$\begin{array}{rc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline & 7+7+7+7+7+7 \end{array} \quad = \; 6 \cdot 7$$

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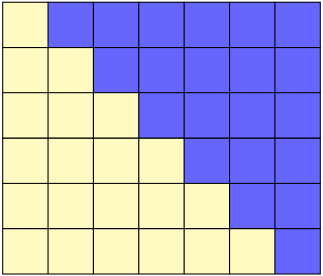
$$\begin{array}{cc} & 1+2+3+4+5+6 \\ + & 6+5+4+3+2+1 \\ \hline & 7+7+7+7+7+7 \end{array} \quad = \; 6 \cdot 7$$

Therefore,

$$1+2+3+4+5+6 = \frac{6 \cdot 7}{2}.$$

An interesting sum

A proof by picture:



$$
\begin{array}{rl}
 & \color{orange}{1+2+3+4+5+6} \\
+ & \color{blue}{6+5+4+3+2+1} \\
\hline
 & 7+7+7+7+7+7 \quad = \quad 6\cdot 7
\end{array}
$$

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Proof. Do the sum twice:

$$\begin{array}{rccccccccc}
 & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\
 & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\
\hline
+ & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1)
\end{array}$$

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The total is $n \cdot (n+1)$.

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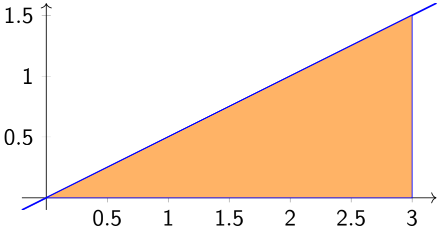
The total is $n \cdot (n+1)$. Divide by two to get the result. $\square$

First integral

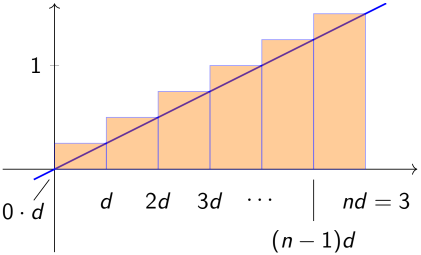
Last time, we were considering a way of finding upper- and lower-bounds for the area under the graph of $f(x) = x/2$ from $x = 0$ to $x = 3$.



Graph of $f(x) = \dfrac{x}{2}$.

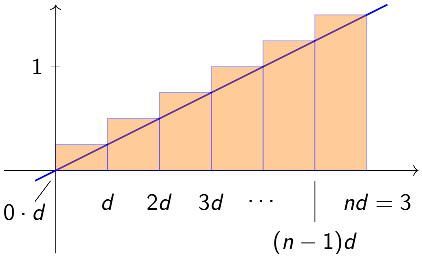
First integral

Divide the interval $[0, 3]$ into n parts, each of size $d = 3/n$.



Graph of $f(x) = \dfrac{x}{2}$.

First integral



Graph of $f(x) = \frac{x}{2}$.

$$\text{sum of areas} = d \cdot \frac{d}{2} + d \cdot \frac{2d}{2} + d \cdot \frac{3d}{2} + \cdot + d \cdot \frac{(nd)}{2}$$

$$= \frac{d^2}{2} \cdot (1 + 2 + 3 + \cdots + n).$$

First integral

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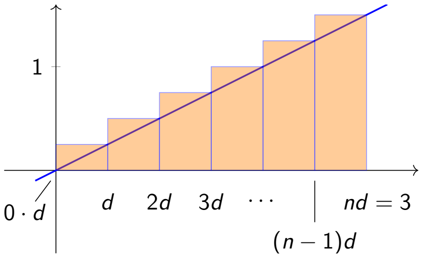
$$= \frac{9}{4} \left(1 + \frac{1}{n}\right).$$

First integral



So some over-estimates of the area of the triangle are $\frac{9}{4}\left(1+\frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$:

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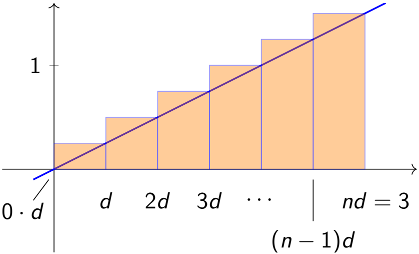

So some over-estimates of the area of the triangle are $\frac{9}{4}\left(1+\frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$:

$$U = \left\{\frac{9}{4}\left(1+\frac{1}{2}\right), \frac{9}{4}\left(1+\frac{1}{3}\right), \frac{9}{4}\left(1+\frac{1}{4}\right), \ldots\right\}.$$

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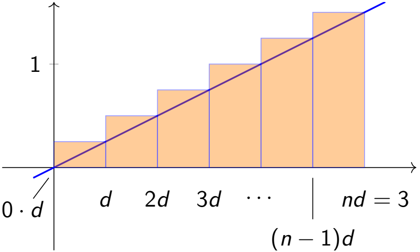

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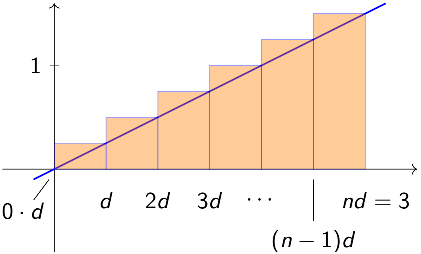
We have

$$\text{area of triangle} \leq \mathrm{glb}(U)$$

First integral



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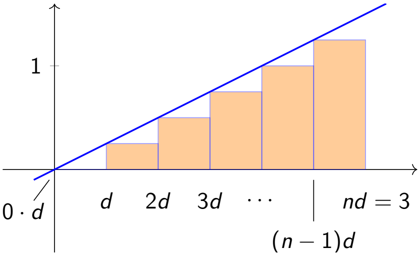
We have

$$\text{area of triangle} \leq \mathrm{glb}(U) = \frac{9}{4}.$$

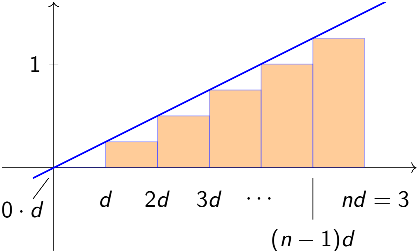
A first integral



Graph of $f(x) = \dfrac{x}{2}$.

Now consider underestimates.

A first integral



Graph of $f(x) = \frac{x}{2}$.

Now consider underestimates.

$$\text{sum of areas} = d \cdot \frac{0 \cdot d}{2} + d \cdot \frac{1 \cdot d}{2} + d \cdot \frac{2d}{2} + \cdot + d \cdot \frac{(n-1)d}{2}$$

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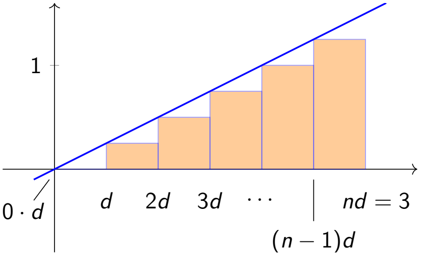
First integral



So some under-estimates of the area of the triangle are $\frac{9}{4}\left(1 - \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$:

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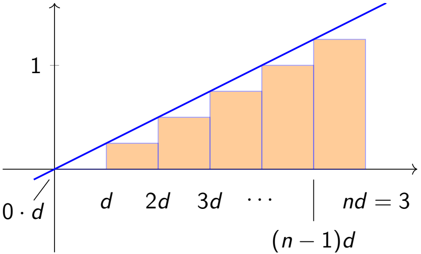

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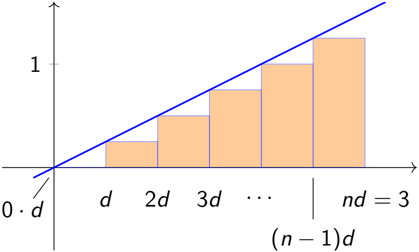
First integral



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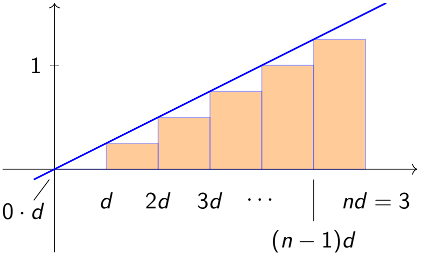
We have

$$\text{area of triangle} \geq \operatorname{lub}(L)$$

First integral



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A first integral

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$$\frac{9}{4} = \mathrm{glb}(L) \leq \text{ area of triangle } \leq \mathrm{lub}(U) = \frac{9}{4}.$$

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$$\frac{9}{4} = \mathrm{glb}(L) \leq \text{ area of triangle } \leq \mathrm{lub}(U) = \frac{9}{4}.$$

Therefore, the area of the triangle is $\frac{9}{4}$.

A first integral

Another argument: we showed that for each $n = 1, 2, 3, \ldots$,

$$\frac{9}{4}\left(1 - \frac{1}{n}\right) \leq \text{ area of triangle } \leq \frac{9}{4}\left(1 + \frac{1}{n}\right).$$

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$$\frac{9}{4}\left(1 - \frac{1}{n}\right) \leq \text{ area of triangle } \leq \frac{9}{4}\left(1 + \frac{1}{n}\right).$$

Since these relations hold for all n, it follows that the area of triangle is $\frac{9}{4}$.