Math 111

October 10, 2022

Today

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- Optimization practice
- Related rates practice

Optimization

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Optimization

Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

Procedure for optimization. Suppose that f is a *continuous* function on a *closed bounded* interval $[a, b]$. Then the (global) minima and maxima for f occur among the following points:

(i) The points in (a, b) at which the derivative of f is 0.

(ii) The points in $[a, b]$ at which f is not differentiable.

(iii) The endpoints, a and b.

Example

Let $f(x) = x^3 - x$. Find the minima and maxima for f on the interval $[-1, 2]$.

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- Endpoints: -1 and 2.

Example

Let $f(x) = x^3 - x$. Find the minima and maxima for f on the interval $[-1, 2]$.

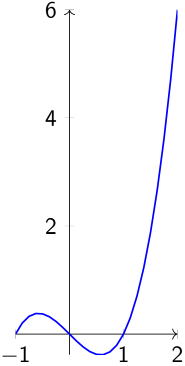
- Continuous on a closed compact interval?
- Points where f is not differentiable on $(-1, 2)$? None.
- Where is $f' = 0$? Answer: $\pm\sqrt{1/3}$.
- Endpoints: -1 and 2.

Evaluate:

$$f\left(-\sqrt{\frac{1}{3}}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.38, \qquad f\left(\sqrt{\frac{1}{3}}\right) = -\frac{2}{3}\sqrt{\frac{1}{3}} \approx -0.38,$$

$$f(-1) = 0, \qquad f(2) = 6.$$

Example

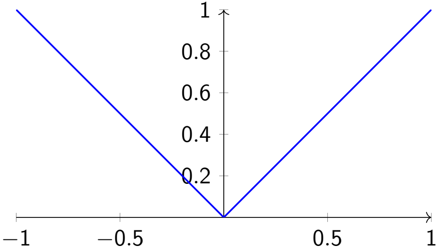


Graph of $f(x) = x^3 - x$ on the interval $[-1, 2]$.

$$f\left(-\sqrt{\frac{1}{3}}\right) \approx 0.38, \quad f\left(\sqrt{\frac{1}{3}}\right) \approx -0.38, \quad f(-1) = 0, \quad f(2) = 6.$$

Example

This example illustrates the fact that in our procedure, we need to check points at which the derivative does not exist. Let $f(x) = |x|$ on the interval $[-1, 1]$:



Graph of $f(x) = |x|$ on the interval $[-1, 1]$.

Example

Consider the function

$$f(x) = \frac{1}{1 - x^2}$$

on the interval $(-1, 1)$.

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What to do? It's still true that a maximum or minimum in the interior of an interval will occur at a place where the derivative is 0.

Check that $f'(x) = \frac{2x}{(1-x^2)^2}$.

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What to do? It's still true that a maximum or minimum in the interior of an interval will occur at a place where the derivative is 0.

Check that $f'(x) = \frac{2x}{(1-x^2)^2}$. So $f' = 0$ only at $x = 0$.

Example

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Global maxima?

Example

Consider the function $f(x) = \frac{1}{1-x^2}$ on the interval $(-1, 1)$.

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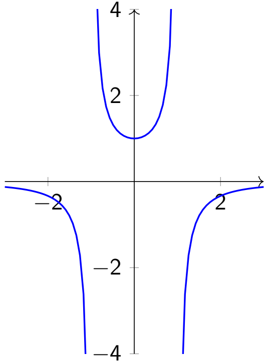
So $f'(x) = 0$ if and only if $x = 0$. Further, note that $f'(x) < 0$ for $-1 < x < 0$ and $f'(x) > 0$ for $0 < x < 1$. so $f(x)$ is decreasing for $-1 < x < 0$ and increasing for $0 < x < 1$.

Therefore, f has a global minimum of $f(0) = 1$ at $x = 0$ on the interval $(-1, 1)$.

Global maxima? None ($\lim_{x\to -1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = \infty$).

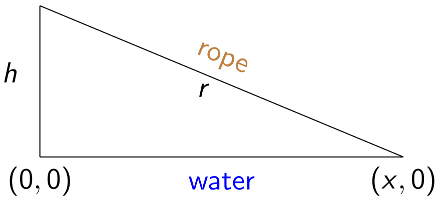
Example



Graph of $f(x) = \frac{1}{1-x^2}$.

Related rates example

Suppose you are standing on a dock pulling in a boat attached to a rope. If you pull the rope in at a constant rate, how does the speed at which the boat approaches the dock change?



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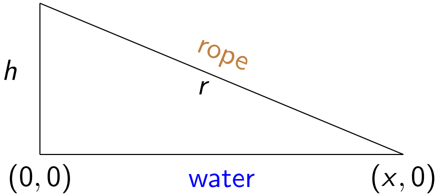

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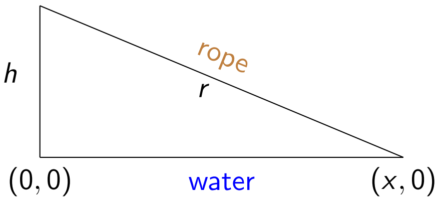

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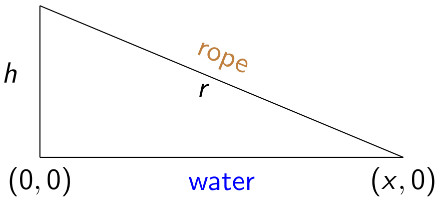

Given $\frac{dr}{dt} = -k =$ constant, find $\frac{dx}{dt}$.

Relation: $x^2 + h^2 = r^2$.

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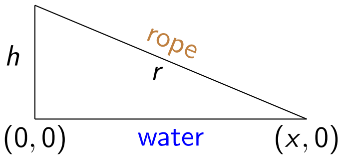

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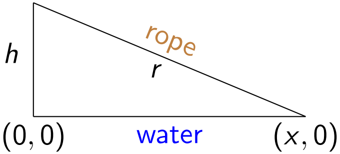

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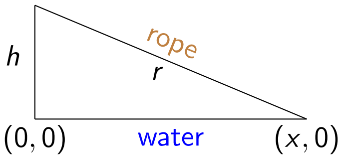

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Chain rule:

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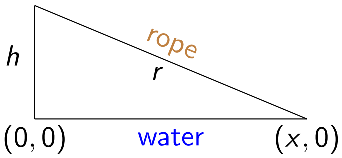

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Chain rule:

$$\frac{d}{dt}(x^2 + h^2) = \frac{d}{dt}(r^2)$$

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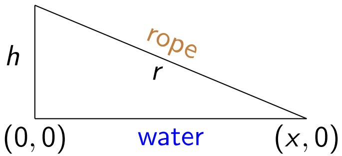

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$$\frac{d}{dt}(x^2 + h^2) = \frac{d}{dt}(r^2) \Rightarrow 2x\frac{dx}{dt} = 2r\frac{dr}{dt}$$

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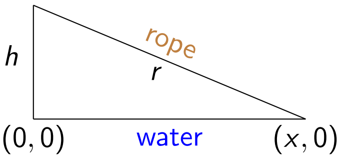

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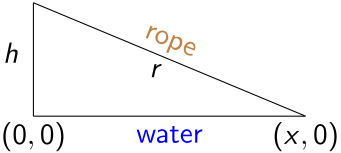

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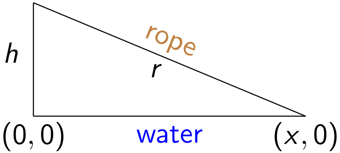

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Therefore,

$$\frac{dx}{dt} = -k\frac{r}{\sqrt{r^2 - h^2}}.$$

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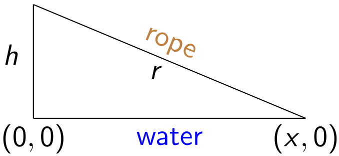

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What happens over time?

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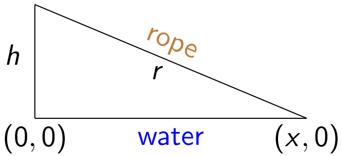

$$\frac{dx}{dt} = -k\frac{r}{\sqrt{r^2 - h^2}}.$$

What happens over time? We have $r \searrow h$, and $\frac{dx}{dt} \to \infty$.