Math 111

October 14, 2022

Today

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- The second derivative test.
- Curve sketching.

Meaning of the second derivative

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and

$$f''(x) = 12x^2 - 6.$$

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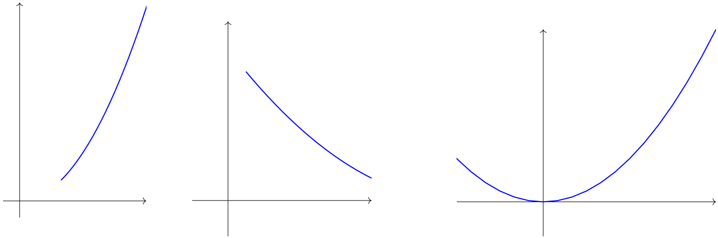

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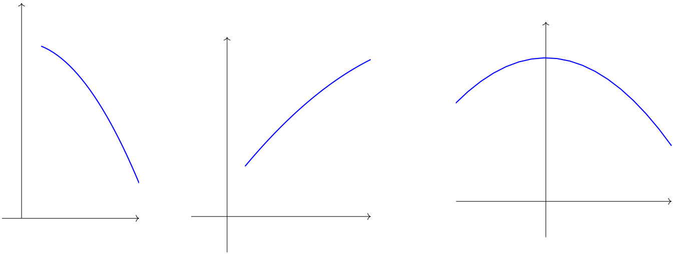

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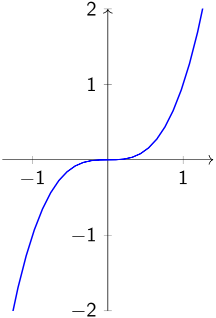
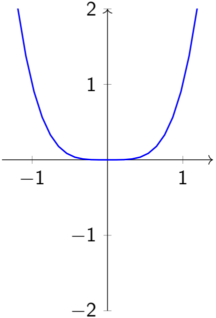
$$f'(c) = 0 \quad \text{and} \quad f''(c) = 0 \implies \text{inconclusive}$$

Second derivative test

Why $f'(c) = f''(c) = 0$ is inconclusive:



$f(x) = x^3$



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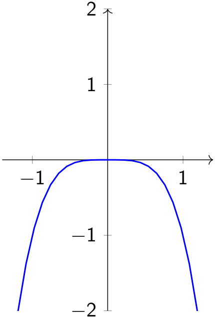

$f(x) = -x^4$

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In the problematic case, where $f'(c) = f''(c) = 0$, use this strategy:

- If there is an interval about c on which $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c. Then f has a local minimum at c.
- If there is an interval about c on which $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c. Then f has a local maximum at c.

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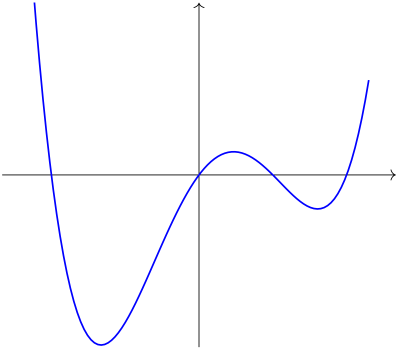
- If there is an interval about c on which $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c. Then f has a local maximum at c.
- If there is an interval about c on which f' has the same sign on either side of c. Then c is a *point of inflection* for f.

Exercise

For the function pictured below, what are the signs of $f'(x)$ and $f''(x)$ at each point x?



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