Math 111

October 5, 2022

Goals

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- **Theorem 2.** (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

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(5, \infty) &= \{x \in \mathbb{R} : x > 5\} \\
[5, \infty) &= \{x \in \mathbb{R} : x \geq 5\} \\
(-\infty, \infty) &= \mathbb{R} = \text{the real numbers}
\end{aligned}$$

Open and closed intervals

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where a and b are real numbers.

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$$(-4,8].$$

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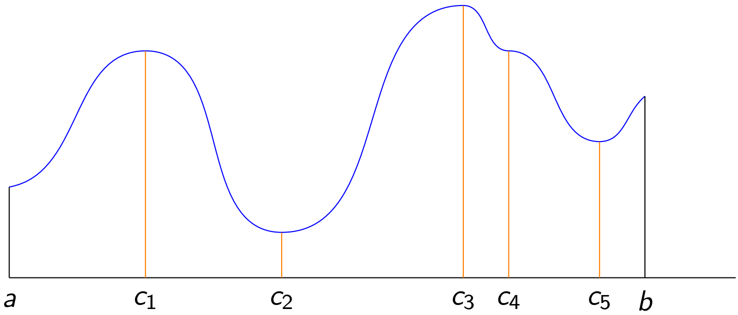

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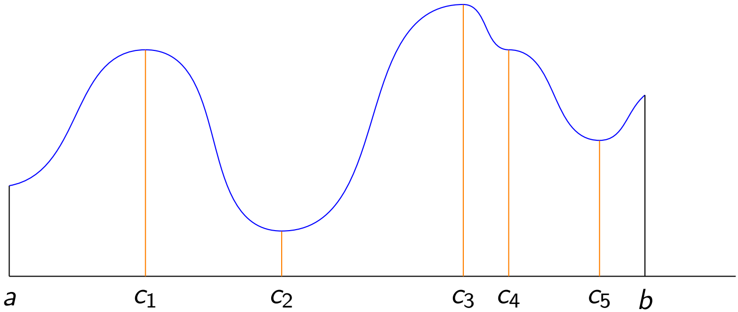

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Answer. The function f has a local maximum at c if and only if there is an open interval containing c and contained in I.

Local minima and maxima

Question. Suppose f is a function defined on an interval I, and suppose that f has a maximum at $c \in I$.

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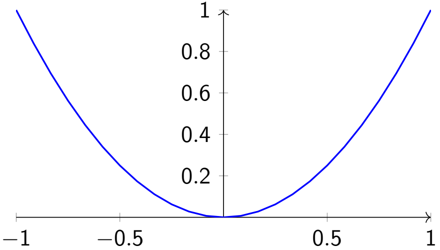
A similar statement holds for minima.

Example



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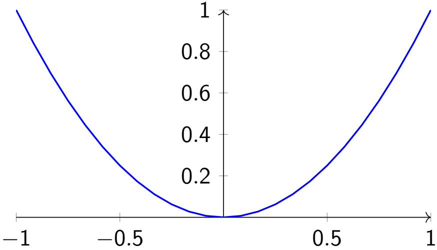

This function defined on $[-1, 1]$, has maxima at ± 1 and a minimum at 0.

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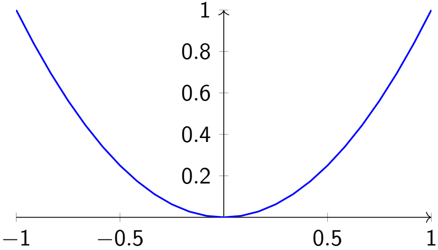

This function defined on $[-1, 1]$, has maxima at ± 1 and a minimum at 0.

It has a local minimum at 0 and no local maxima.

Example

Question. Let $f(x) = 1$ for all real numbers x. What are the global and local maxima and minima of f?

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Proof. See board (or notes).

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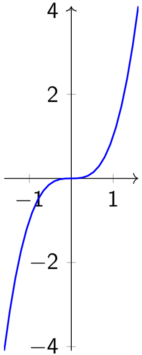
Answer: No. Consider $f(x) = x^3$ at $x = 0$:



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Let $f(x) = x^3 - x^2$. What are the minima and maxima of f, both local and global, on $I = [-1/2, 1/2]$?

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- We need to check the points where $f' = 0$ and the endpoints of the interval, ± 1.

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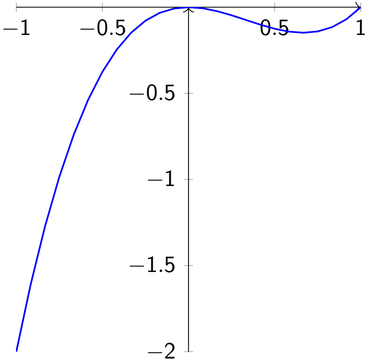

Graph of $f(x) = x^3 - x^2$.

Note the local minumum at $x = 2/3$.