Math 111

October 7, 2022

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- Appreciate the hypotheses in the Extreme Value Theorem (EVT).
- Outline procedure for optimization + examples.

Proof of derivative theorem for local extrema

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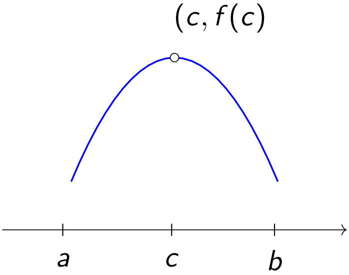

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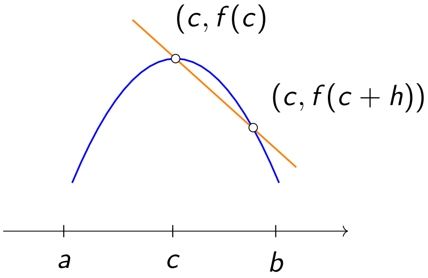

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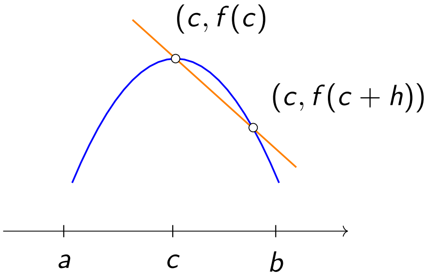

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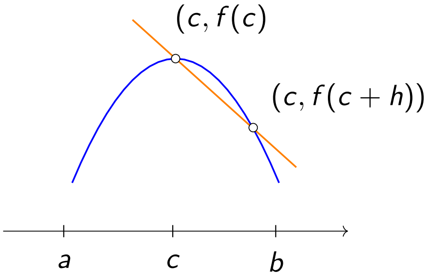

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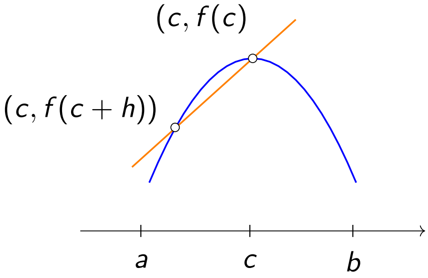

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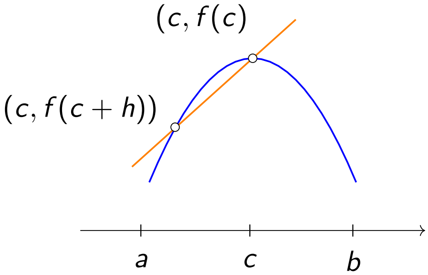

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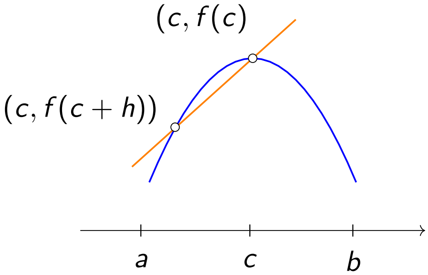

$$f(c+h) - f(c) \leq 0 \quad \text{and} \quad h < 0 \quad \implies \quad \frac{f(c+h) - f(c)}{h} \geq 0$$

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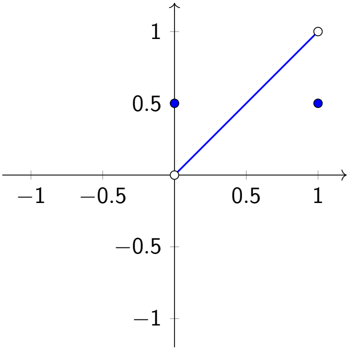
Extreme value theorem

Analyze the following function with regards to the EVT:

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = \pm 1. \end{cases}$$



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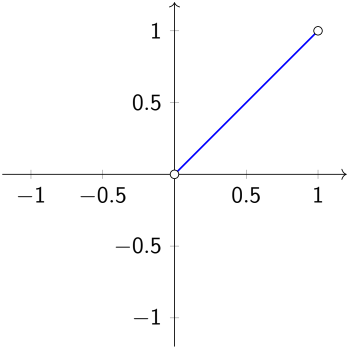
$$\begin{aligned} g\colon (0,1) &\to \mathbb{R} \\ x &\mapsto x. \end{aligned}$$



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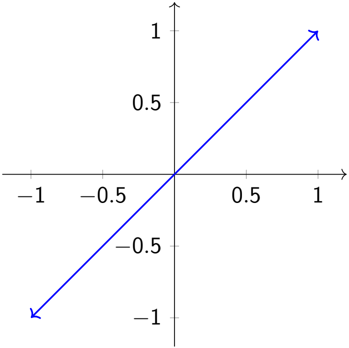
Analyze the following function with regards to the EVT:

$$h\colon (-\infty, \infty) \to \mathbb{R}$$
$$x \mapsto x.$$



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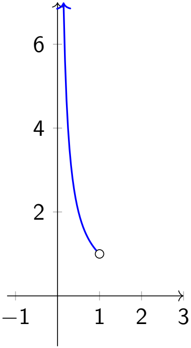
Analyze the following function with regards to the EVT:

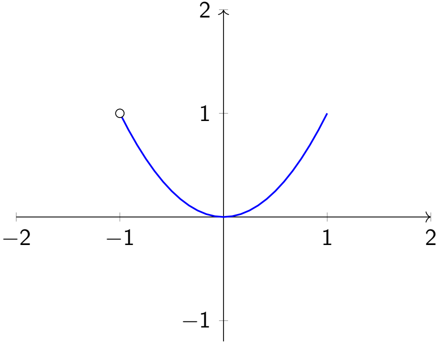
$$\begin{aligned} k\colon (0,1) &\to \mathbb{R} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$



Extreme value theorem

Analyze the following function with regards to the EVT:

$$\begin{aligned} \ell \colon (-1, 1] &\to \mathbb{R} \\ x &\mapsto x^2. \end{aligned}$$



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Theorem 1. If f is differentiable at c and f has a local minimum or maximum at c, then $f'(c) = 0$.

Theorem 2. (The extreme value theorem, EVT) If f is continuous on a closed bounded interval $[a, b]$, then f has a (global) minimum and maximum on that interval.

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Procedure for optimization. Suppose that f is a continuous function on a *closed bounded* interval $[a, b]$. Then the (global) minima and maxima for f occur among the following points:

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The points satisfying 1 or 2 are called *critical points*. Evaluate f at the critical points and the endpoints. The smallest value will give the minimum and the largest will give the maximum.

Example

Find the extrema, local and global, for

$$f(x) = x^3 - x^2$$

on $I = [-1/2, 1/2]$.

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- We have $f(2/3) = -4/27$, but $2/3 \notin [-1/2, 1/2]$.
- So on the interval $[-1/2, 1/2]$, the function f has minimum at $-1/2$ and a maximum at 0.

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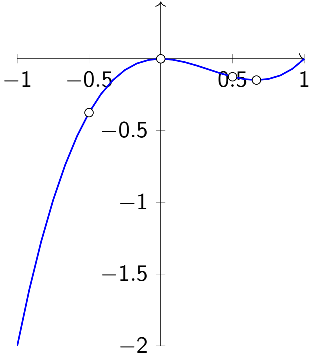

Graph of $f(x) = x^3 - x^2$.

Note the local minimum at $x = 2/3$.

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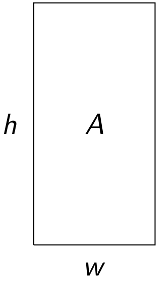
Suppose a farmer has ℓ feet of fence and wants to make a rectangular enclosure of the maximal area. What should the dimensions of the rectangle be?



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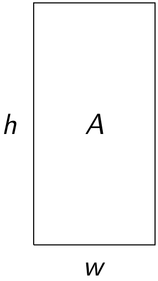

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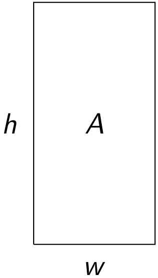

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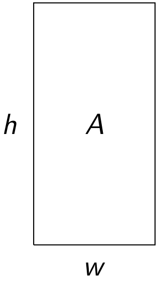

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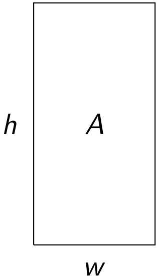

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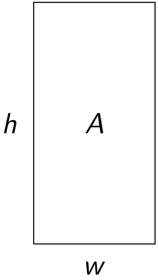

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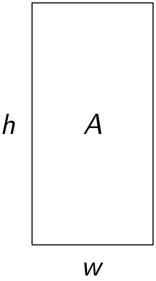

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$w \in [0, \ell/2]$ (closed and bounded)

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The only other place a maximum could occur is at the endpoints, 0 and $\ell/2$. So we evaluate A at the critical point and at the endpoints, to see which points maximizes A. (See next page.)

Example

We have $A = \frac{1}{2}\ell w - w^2$. So

$$A(0) = 0, \quad A(\ell/2) = 0, \quad \text{and} \quad A(\ell/4) = \frac{\ell^2}{16}.$$

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Recall that $\ell = 2w + 2h$.

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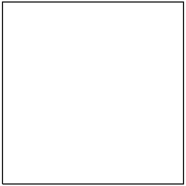
Recall that $\ell = 2w + 2h$. So when $w = \ell/4$, it follows that $h = \ell/4$, too.

Example

We have $A = \frac{1}{2}\ell w - w^2$. So

$$A(0) = 0, \quad A(\ell/2) = 0, \quad \text{and} \quad A(\ell/4) = \frac{\ell^2}{16}.$$

Recall that $\ell = 2w + 2h$. So when $w = \ell/4$, it follows that $h = \ell/4$, too. That means the area of a rectangle with fixed perimeter is maximized when it is a square:



$h = \ell/4$

$w = \ell/4$