Math 111

September 28, 2022

Goals

- Different notation for chain rule.
- Tangent lines for implicitly defined functions.

Chain rule

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$$(e^{\cos(x)})' = e^{\cos(x)}(-\sin(x)) = -\sin(x)e^{\cos(x)}.$$

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Nice notation for the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2) \cdot (2t) = (3(t^2 + 4)^2)(2t).$$

Implicit functions, example 1

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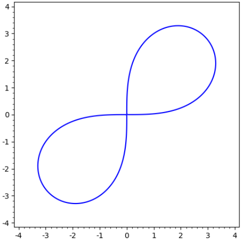

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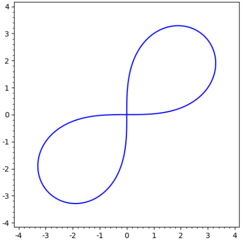

We say that y is defined *implicitly* as a function of x by the above equation.

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Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{13}{9}.$$

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The equation of the tangent line to the curve defined by

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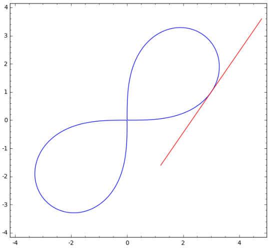

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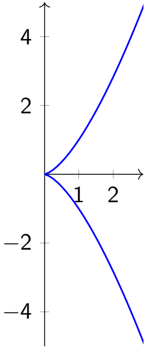

The cuspidal cubic curve, $y^2 = x^3$.

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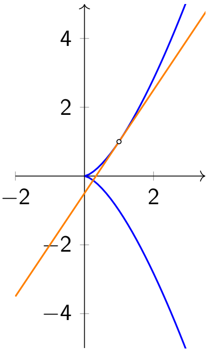
The cuspidal cubic $y^2 = x^3$ with its tangent line $y = 1 + \frac{3}{2}(x - 1)$ at the point $(1, 1)$:



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The point $(1,1)$ sits on the branch $y = x^{3/2}$. Take the derivative to find the slope:

$$y' = \frac{3}{2}x^{1/2}.$$

At $(1,1)$, the slope is $3/2$, as we found before.

Related rates

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3. Use the chain rule to differentiate the equation relating the variables.
4. Substitute into this equation all the quantities we know and solve for the quantity we are trying to determine.