Math 111

September 26, 2022

Goals

Today's goal: the chain rule.

What is the derivative of the composition of two differentiable functions?

Derivative theorem

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Composition rule: If f and g are continuous functions, then

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(g(c)).$$

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Note that

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$$f(x) = \frac{1}{x^{17}} \quad \text{and} \quad g(x) = 5x^2 + x.$$

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For multiple compositions, we can apply the chain rule multiple times:

$$(f(g(h(x))))' = f'(g(h(x)))g'(h(x))h'(x).$$

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$$\begin{aligned}(\sin((x^4 + 5x - 2)^{10}))' \\ &= \cos((x^4 + 5x - 2)^{10})(10(x^4 + 5x - 2)^9)(4x^3 + 5) \\ &= 10(x^4 + 5x - 2)^9(4x^3 + 5)\cos((x^4 + 5x - 2)^{10}).\end{aligned}$$

$$f(x) = \sin(x), \quad g(x) = x^{10}, \quad \text{and} \quad h(x) = x^4 + 5x - 2.$$

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&= 2x \sin(\ln(x)) + x^2 \cos(\ln(x))(1/x)
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&= 2x \sin(\ln(x)) + x \cos(\ln(x)).
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