# Math 111

September 19, 2022

# Goals

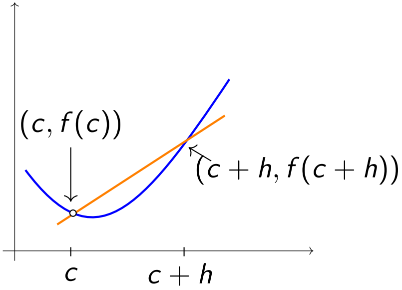
- Use the definition to calculate the derivative.
- Find the equation of the tangent line to a function at a given point.

# The derivative (instantaneous rate of change)

**Definition.** The *derivative* of the function $f$ at the point $c$ is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists.

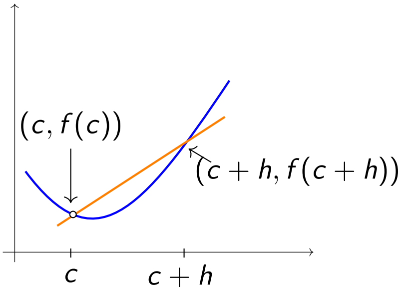



The slope of the orange *secant* line is $\frac{f(c+h)-f(c)}{h}$.

# Average versus instantaneous rate of change

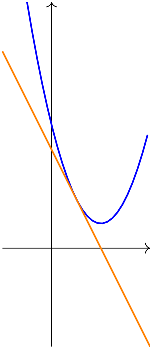
Average rate of change: $\dfrac{f(c+h)-f(c)}{h}$

Instantaneous rate of change: $f'(c) = \lim_{h \to 0} \dfrac{f(c+h)-f(c)}{h}$



# Synonyms

$$\begin{aligned} f'(c) &= \text{derivative at } c \\ &= \text{instantaneous rate of change of } f \text{ at } c \\ &= \text{slope of } f \text{ at } c \end{aligned}$$



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&= \lim_{h\to 0} \frac{6h+h^2}{h} \\
&= \lim_{h\to 0} (6+h) \\
&= 6.
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&= 2x.
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Compute the derivative of $f(x) = x^4$ at an arbitrary point $x$.

Solution: $f'(x) = 4x^3$. (Work done on the blackboard.)

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Solution: $f'(x) = nx^{n-1}$. (Work done on the blackboard.)

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Solution: $f'(x) = m$. (Work done on the blackboard.)

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The tangent line has equation $y = 6x - 9$.

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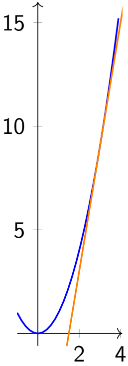


Graph of $f(x) = x^2$ and its tangent line at $x = 3$.

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Solution: We have $f'(x) = 3x^2$. So $f'(1) = 3 \cdot 2^2 = 12$. The line has equation $y = 12x + b$ for some $b$. Since the line passes through $(2, f(2)) = (2, 8)$,

$$8 = 12 \cdot 2 + b.$$

Therefore, $b = -16$, and the tangent line at $x = 2$ is

$$y = 12x - 16.$$