Math 111

September 23, 2022

Goals

- Prove our derivative theorem for combining simple functions to make complicated functions.
- Use the theorem to compute some derivatives of specific functions.

Limit theorem

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2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$,
3. if $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Derivative theorem

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2. *product rule or Leibniz rule*:
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3. *quotient rule*:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$$

Proof of sum formula for derivatives

Claim. If f and g are differentiable at x, then

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$$(f(x)+g(x))' = \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

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&= f'(x)+g'(x).
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&= \lim_{h\to 0} \left(\frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h} \right)
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&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

Proof of product rule for derivatives (continued)

Continuing from where we were:

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Proof of product rule for derivatives (continued)

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$$(f(x)g(x))' = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

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Proof of product rule for derivatives (continued)

Continuing from where we were:

$$\begin{aligned}(f(x)g(x))' &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)\lim_{h\to 0} g(x+h) + (\lim_{h\to 0} f(x))g'(x)\end{aligned}$$

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Proof of product rule for derivatives (continued)

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\end{aligned}
$$

We are done if we can show that

$$\lim_{h\to 0} f(x) = f(x)$$

and

$$\lim_{h\to 0} g(x+h) = g(x).$$

Proof of product rule for derivatives (continued)

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As far $\lim_{h \to 0}$ is concerned, $f(x)$ is a constant, and the limit of a constant function is the constant, itself.

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Proposition. Differentiable functions are continuous.

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Leave x fixed, and define a function of h by $k(h) = x + h$.

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Leave x fixed, and define a function of h by $k(h) = x + h$. Since the composition of continuous functions is continuous, the following function is continuous:

$$(g \circ k)(h) = g(k(h)) = g(x+h).$$

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Leave x fixed, and define a function of h by $k(h) = x + h$. Since the composition of continuous functions is continuous, the following function is continuous:

$$(g \circ k)(h) = g(k(h)) = g(x+h).$$

Therefore,

$$\lim_{h \to 0} g(x+h) = g(x).$$

Proof of the quotient rule

For a proof of the quotient rule,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

see the lecture notes or our text.

Example of use of our derivative theorem

$$\left(3x^4 + x^2 - 4x + 2\right)'$$

Example of use of our derivative theorem

$$\left(3x^4 + x^2 - 4x + 2\right)' = (3x^4)' + (x^2)' + (-4x)' + (2)'$$

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$$\begin{aligned}\left(3x^4 + x^2 - 4x + 2\right)' &= (3x^4)' + (x^2)' + (-4x)' + (2)' \\ &= 3(x^4)' + (x^2)' - 4(x)' + 2(1)'\end{aligned}$$

Example of use of our derivative theorem

$$\begin{aligned}
\left(3x^4 + x^2 - 4x + 2\right)' &= (3x^4)' + (x^2)' + (-4x)' + (2)' \\
&= 3(x^4)' + (x^2)' - 4(x)' + 2(1)' \\
&= 3(4x^3) + (2x) - 4(1) + 2(0)
\end{aligned}$$

Example of use of our derivative theorem

$$\begin{aligned}
\left(3x^4 + x^2 - 4x + 2\right)' &= (3x^4)' + (x^2)' + (-4x)' + (2)' \\
&= 3(x^4)' + (x^2)' - 4(x)' + 2(1)' \\
&= 3(4x^3) + (2x) - 4(1) + 2(0) \\
&= 12x^3 + 2x - 4.
\end{aligned}$$

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$$(6x^5 - 4x^3 + 12x^2 - 7x + 2)' =$$

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$$(6x^5 - 4x^3 + 12x^2 - 7x + 2)' = 30x^4 - 12x^2 + 24x - 7.$$

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Example of use of our derivative theorem

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In general, $n = 1, 2, 3, \ldots$,

$$(x^{-n})' = \left(\tfrac{1}{x^n}\right)' = \tfrac{(1)'x^n - 1\cdot(x^n)'}{(x^n)^2} = -\tfrac{(x^n)'}{(x^n)^2} = -\tfrac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

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In general, $n = 1, 2, 3, \ldots$,

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{(1)'x^n - 1 \cdot (x^n)'}{(x^n)^2} = -\frac{(x^n)'}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

Therefore,

$$x^n = nx^{n-1}$$

for $n = 0, \pm 1, \pm 2, \ldots$

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$$\left(\frac{x^2}{x^4+3x+2}\right)' = \frac{(x^2)'(x^4+3x+2) - x^2(x^4+3x+2)'}{(x^4+3x+2)^2}$$

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$$= \frac{2x(x^4+3x+2)-x^2(4x^3+3)}{(x^4+3x+2)^2}$$

$$= \frac{2x^5+6x^2+4x-4x^5-3x^2}{(x^4+3x+2)^2}$$

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&= \frac{2x(x^4+3x+2) - x^2(4x^3+3)}{(x^4+3x+2)^2} \\
&= \frac{2x^5+6x^2+4x-4x^5-3x^2}{(x^4+3x+2)^2} \\
&= \frac{-2x^5+3x^2+4x}{(x^4+3x+2)^2}.
\end{aligned}$$