# Math 111

September 14, 2022

# Continuity

**Definition.** The function $f$ is *continuous at a point* $c \in \mathbb{R}$ if

$$\lim_{x \to c} f(x) = f(c).$$

If $f$ is continuous at every point, we simply say $f$ is a *continuous* function.

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

Compute, using our limit theorems:

$$\lim_{x \to 2} (5x^2 + 1)$$

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

Compute, using our limit theorems:

$$\lim_{x \to 2} (5x^2 + 1) = \lim_{x \to 2} 5x^2 + \lim_{x \to 2} 1$$

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

Compute, using our limit theorems:

$$\begin{aligned} \lim_{x \to 2} (5x^2 + 1) &= \lim_{x \to 2} 5x^2 + \lim_{x \to 2} 1 \\ &= \lim_{x \to 2} 5 \lim_{x \to 2} x \lim_{x \to 2} x + \lim_{x \to 2} 1 \end{aligned}$$

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

Compute, using our limit theorems:

$$\begin{aligned}
\lim_{x\to 2}(5x^2 + 1) &= \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 1 \\
&= \lim_{x\to 2} 5 \lim_{x\to 2} x \lim_{x\to 2} x + \lim_{x\to 2} 1 \\
&= 5 \cdot 2 \cdot 2 + 1 = 21.
\end{aligned}$$

# Example

Show $f(x) = 5x^2 + 1$ is continuous at $c = 2$.

Compute, using our limit theorems:

$$\begin{aligned}\lim_{x \to 2}(5x^2 + 1) &= \lim_{x \to 2} 5x^2 + \lim_{x \to 2} 1 \\ &= \lim_{x \to 2} 5 \lim_{x \to 2} x \lim_{x \to 2} x + \lim_{x \to 2} 1 \\ &= 5 \cdot 2 \cdot 2 + 1 = 21.\end{aligned}$$

Thus, $\lim_{x \to 2} f(x) = f(c)$, as required.

# Example.

Show $f(x) = \frac{3x^2-5}{x^3-2x+3}$ is continuous at $c = 1$.

# Example.

Show $f(x) = \frac{3x^2-5}{x^3-2x+3}$ is continuous at $c = 1$.

Compute, using our limit theorems:

$$\lim_{x \to 1} \frac{3x^2 - 5}{x^3 - 2x + 3} =$$

.

# Example.

Show $f(x) = \frac{3x^2-5}{x^3-2x+3}$ is continuous at $c = 1$.

Compute, using our limit theorems:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3) \lim_{x\to 1} x \lim_{x\to 1} x + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3}$$

.

# Example.

Show $f(x) = \frac{3x^2-5}{x^3-2x+3}$ is continuous at $c = 1$.

Compute, using our limit theorems:

$$\lim_{x\to 1} \frac{3x^2 - 5}{x^3 - 2x + 3} = \frac{\lim_{x\to 1}(3)\lim_{x\to 1} x \lim_{x\to 1} x + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3}$$

$$= \frac{3(1\cdot 1) - 5}{1\cdot 1\cdot 1 - 2\cdot 1 + 3} = \frac{-2}{2} = -1.$$

# Example.

Show $f(x) = \frac{3x^2-5}{x^3-2x+3}$ is continuous at $c = 1$.

Compute, using our limit theorems:

$$\lim_{x\to 1} \frac{3x^2 - 5}{x^3 - 2x + 3} = \frac{\lim_{x\to 1}(3) \lim_{x\to 1} x \lim_{x\to 1} x + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3}$$

$$= \frac{3(1\cdot 1) - 5}{1\cdot 1\cdot 1 - 2\cdot 1 + 3} = \frac{-2}{2} = -1.$$

Thus, $\lim_{x\to 1} f(x) = f(c)$, as required.

# Rational functions are continuous

Extrapolating from the previous examples, if $f$ and $g$ are polynomials, then $\frac{f}{g}$ is continuous at any point $c$ such that $g(c) \neq 0$.

# Rational functions are continuous

Extrapolating from the previous examples, if $f$ and $g$ are polynomials, then $\frac{f}{g}$ is continuous at any point $c$ such that $g(c) \neq 0$.

For instance,

$$\frac{3x^5 - 7x^2 + 4x + 1}{x^2 - 4}$$

is continuous at all real numbers except 2 and $-2$.

# More examples of continuous functions

Most of the functions encountered in high school are continuou wherever they are defined:

$$\sqrt{x},\ \ e^x,\ \ \ln(x), \cos(x), \sin(x), \tan(x), \text{etc.}$$

# Example of function that is not continuous

The following function is not continuous at 0.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

# Example of function that is not continuous

The following function is not continuous at 0.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Reason:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$$

# Example of function that is not continuous

The following function is not continuous at 0.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Reason:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0) = 1.$$

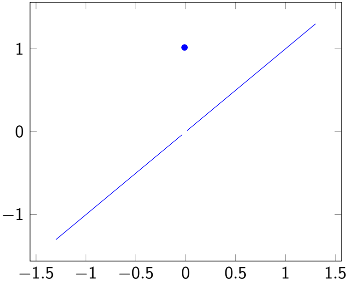
# Example of function that is not continuous

The following function is not continuous at 0.

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Reason:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0) = 1.$$



# Example

The function

$$f(h) = \frac{h(h-2)}{h}$$

is not continuous at $h = 0$.

# Example

The function

$$f(h) = \frac{h(h-2)}{h}$$

is not continuous at $h = 0$.

If $f$ is continuous at $c$, then $f$ must be defined at $c$. In other words, $f(c)$ must exist.

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 3) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3},$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = \sqrt{x + 3},$$

and

$$(g \circ f)(x) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x + 3) = \sqrt{x + 3},$$

and

$$(g \circ f)(x) = g(f(x)) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3},$$

and

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) =$$

# Compositions of functions

Recall that if $f(x)$ and $g(x)$ are functions, then the *composition* of $f$ and $g$ is the function:

$$(f \circ g)(x) := f(g(x)).$$

**Example.** If $f(x) = \sqrt{x}$ and $g(x) = x + 3$, then

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3},$$

and

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 3.$$

# Composition of continuous functions

**Composition theorem.** If $f$ and $g$ are continuous functions, then

$$\lim_{x \to c} (f \circ g)(x) = f(g(c)).$$

# Composition of continuous functions

**Composition theorem.** If $f$ and $g$ are continuous functions, then

$$\lim_{x \to c} (f \circ g)(x) = f(g(c)).$$

**Proof.** Math 112. $\square$

# Composition of continuous functions

**Composition theorem.** If $f$ and $g$ are continuous functions, then

$$\lim_{x \to c} (f \circ g)(x) = f(g(c)).$$

**Proof.** Math 112. $\square$

Mantra: The composition of continuous functions is continuous.

# Example of composition theorem in action.

**Claim.** $\lim_{x \to 6} \sqrt{x+3} = 3$ (used in the previous lecture).

# Example of composition theorem in action.

**Claim.** $\lim_{x \to 6} \sqrt{x+3} = 3$ (used in the previous lecture).

Let $f(x) = \sqrt{x}$ and $g(x) = x + 3$.

# Example of composition theorem in action.

**Claim.** $\lim_{x \to 6} \sqrt{x+3} = 3$ (used in the previous lecture).

Let $f(x) = \sqrt{x}$ and $g(x) = x + 3$. Since $f$ and $g$ are continuous, so is $f \circ g$.

# Example of composition theorem in action.

**Claim.** $\lim_{x \to 6} \sqrt{x+3} = 3$ (used in the previous lecture).

Let $f(x) = \sqrt{x}$ and $g(x) = x + 3$. Since $f$ and $g$ are continuous, so is $f \circ g$. In other words,

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3}$$

is continuous.

# Example of composition theorem in action.

**Claim.** $\lim_{x \to 6} \sqrt{x+3} = 3$ (used in the previous lecture).

Let $f(x) = \sqrt{x}$ and $g(x) = x + 3$. Since $f$ and $g$ are continuous, so is $f \circ g$. In other words,

$$(f \circ g)(x) = f(g(x)) = f(x+3) = \sqrt{x+3}$$

is continuous. The result follows.