Math 111

September 12, 2022

Limit Theorem

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(c) If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

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Main point. The limit allows us to construct limits of complicated functions from limits of simpler functions.

Example

$$\lim_{x \to 2} (5x^2 + 3x + 1)$$

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$$= (\lim_{x \to 2} 5)(\lim_{x \to 2} x)(\lim_{x \to 2} x) + (\lim_{x \to 2} 3)(\lim_{x \to 2} x) + \lim_{x \to 2} 1$$

Example

$$\begin{aligned}\lim_{x\to 2}(5x^2+3x+1) &= \lim_{x\to 2} 5x^2 + \lim_{x\to 2} 3x + \lim_{x\to 2} 1 \\ &= (\lim_{x\to 2} 5)(\lim_{x\to 2} x)(\lim_{x\to 2} x) + (\lim_{x\to 2} 3)(\lim_{x\to 2} x) + \lim_{x\to 2} 1 \\ &= 5\cdot 2\cdot 2 + 3\cdot 2 + 1 = 27.\end{aligned}$$

Some simple limits

Proposition 1. Let c and k be real numbers. Then

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2. $\lim_{x \to c} x = c$.

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Examples.

$$\lim_{x\to 5} 13 = 13$$

$$\lim_{x\to 5} x = 5.$$

Proof of one of the simple limits

Would like to show $\lim_{x \to c} x = c$.

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Let $\varepsilon > 0$ be given.

We must find $\delta > 0$ such that if x satisfies

$$0 < |x - c| < \delta,$$

then

$$|f(x) - L| < \varepsilon.$$

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But $|f(x) - L| = |x - c|$.

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Answer: Take $\delta = \varepsilon$.

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Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon$ and suppose $0 < |x - c| < \delta$.

Then it follows that $|x - c| < \varepsilon$. $\square$.

Example of the Limit Theorem in action

Proposition 2. If $\lim_{x \to c} f(x)$ exists and $k \in \mathbb{R}$, then

$$\lim_{x \to c} k\, f(x) = k \lim_{x \to c} f(x).$$

Example of the Limit Theorem in action

Proposition 2. If $\lim_{x \to c} f(x)$ exists and $k \in \mathbb{R}$, then

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Proof. Using the limit theorem and Proposition 1, we have

$$\lim_{x \to c} k\, f(x) = \lim_{x \to c} k \lim_{x \to c} f(x)$$

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Example of the Limit Theorem in action

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$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} \tag{LTc}$$

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$$= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x^3 + \lim_{x\to 1} -2x + \lim_{x\to 1} 3)} \tag{LTa}$$

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$$= \frac{3(\lim_{x\to1} x^2) - 5}{\lim_{x\to1} x^3 - 2\lim_{x\to1} x + \lim_{x\to1} 3} \tag{Prop 2 and Prop 1}$$

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$$= \frac{3(\lim_{x\to 1} x \lim_{x\to 1} x) - 5}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad \text{(LTb)}$$

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$$= \frac{3(\lim_{x\to 1} x^2) - 5}{\lim_{x\to 1} x^3 - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad \text{(Prop 2 and Prop 1)}$$

$$= \frac{3(\lim_{x\to 1} x \lim_{x\to 1} x) - 5}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad \text{(LTb)}$$

$$= \frac{3(1\cdot 1) - 5}{1\cdot 1\cdot 1 - 2\cdot 1 + 3} = \frac{-2}{2} = -1 \qquad \text{(Prop 1)}$$

Cancellation trick

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$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$$

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Cancellation trick

$$
\begin{aligned}
\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x+3)}{x-2} \\
&= \lim_{x \to 2} (x + 3) \\
&= \lim_{x \to 2} x + \lim_{x \to 2} 3 \\
&= 2 + 3 \\
&= 5.
\end{aligned}
$$

Rationalization trick

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$$\lim_{x \to 6} \frac{\sqrt{x+3}-3}{x-6}$$

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$$\lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6} = \lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6} \cdot \frac{\sqrt{x+3}+3}{\sqrt{x+3}+3}$$

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$$= \lim_{x\to 6}\frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)}$$

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$$= \lim_{x\to 6} \frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)}$$

$$= \lim_{x\to 6} \frac{(x+3)-9}{(x-6)(\sqrt{x+3}+3)}$$

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&= \lim_{x \to 6} \frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{(x+3)-9}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{x-6}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x \to 6} \frac{1}{\sqrt{x+3}+3}
\end{aligned}$$

Rationalization trick

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\lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6} &= \lim_{x\to 6} \frac{\sqrt{x+3}-3}{x-6} \cdot \frac{\sqrt{x+3}+3}{\sqrt{x+3}+3} \\
&= \lim_{x\to 6} \frac{(\sqrt{x+3}-3)(\sqrt{x+3}+3)}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6} \frac{(x+3)-9}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6} \frac{x-6}{(x-6)(\sqrt{x+3}+3)} \\
&= \lim_{x\to 6} \frac{1}{\sqrt{x+3}+3} \\
&= \frac{1}{6}.
\end{aligned}$$

Proof of one part of the limit theorem

We would like to show that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M.$$

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We will need the *triangle inequality*:

$$|a + b| \leq |a| + |b|$$

for all real numbers a, b.

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Example:

$$|-4 + 7| \leq |-4| + |7|.$$

Proof

Let $\varepsilon > 0$ be given.

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Since $\lim_{x \to c} f(x) = L$, there exists $\delta_1 > 0$ such that if x satisfies $0 < |x - c| < \delta_1$, then

$$|f(x) - L| < \frac{\varepsilon}{2}.$$

Proof

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Since $\lim_{x \to c} g(x) = L$, there exists $\delta_2 > 0$ such that if x satisfies $0 < |x - c| < \delta_2$, then

$$|g(x) - M| < \frac{\varepsilon}{2}.$$

Proof

Let $\varepsilon > 0$ be given.

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Since $\lim_{x \to c} g(x) = L$, there exists $\delta_2 > 0$ such that if x satisfies $0 < |x - c| < \delta_2$, then

$$|g(x) - M| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Proof, continued

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$.

Proof, continued

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$.

Therefore,

$\square$

Proof, continued

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$.

Therefore,

$$|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)|$$

$\square$

Proof, continued

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$.

Therefore,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \quad (\triangle\text{-inequality}) \end{aligned}$$

□

Proof, continued

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$.

Therefore,

$$
\begin{aligned}
|(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\
&\leq |f(x) - L| + |g(x) - M| \quad (\triangle\text{-inequality}) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

□