# Math 111

September 7, 2022

# Limits

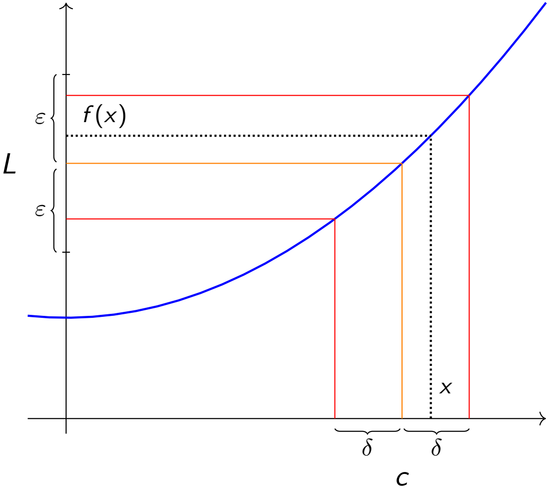
**Definition.** Let $f$ be a function defined in an open interval containing a point $c$, except $f$ might not be defined at the point $c$, itself. Let $L$ be a real number. The *limit of $f(x)$ as $x$ approaches $c$* is $L$, denoted $\lim_{x \to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such if $x$ satisfies

$$0 < |x - c| < \delta,$$

then

$$|f(x) - L| < \varepsilon.$$

# Relevant diagram

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Given $\varepsilon > 0$, let $\delta = \varepsilon/5$. Suppose that $0 < |x-7| < \delta = \varepsilon/5$. Then

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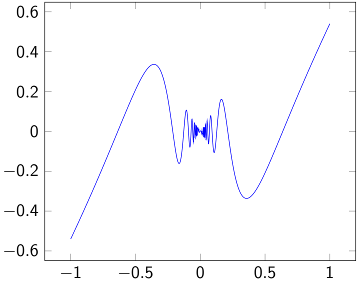


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Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/11\}$, i.e., $\delta$ is the minumum of 1 and $\varepsilon/11$. So $\delta \leq 1$ and $\delta \leq \varepsilon/11$ (with equality holding in at least one of these).

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Since $\delta \leq 1$, it follows that

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Since $\delta \leq 1$, it follows that $4 < x < 6$, and hence $9 < x + 5 < 11$.

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In particular, $|x + 5| < 11$.

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Since $\delta \leq 1$, it follows that $4 < x < 6$, and hence $9 < x + 5 < 11$.
In particular, $|x + 5| < 11$. Therefore,

$$|x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5| < 11|x - 5|.$$

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Since $\delta \leq 1$, it follows that $4 < x < 6$, and hence $9 < x + 5 < 11$.
In particular, $|x + 5| < 11$. Therefore,

$$|x^2 - 25| = |(x+5)(x-5)| = |x+5||x-5| < 11|x-5|.$$

Now, since $\delta \leq \varepsilon/11$ and $|x - 5| < \delta$, it follows that

$$|x^2 - 25| < 11|x-5| < 11 \cdot \frac{\varepsilon}{11} = \varepsilon,$$

as required. $\square$

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2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

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2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.
3. If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

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$$\lim_{x \to 5} x^2 = \lim_{x \to 5} (x \cdot x) = \left( \lim_{x \to 5} x \right) \left( \lim_{x \to 5} x \right) = 5 \cdot 5 = 25.$$

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