Math 111

September 9, 2022

Limit Theorem

Suppose that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Limit Theorem

Suppose that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Then

1. $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

Limit Theorem

Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

Then

1. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.

Limit Theorem

Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

Then

1. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$.
3. If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Limit Theorem

Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

Then

1. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
2. $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x)\, \lim_{x\to c} g(x)$.
3. If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Main point. The limit allows us to construct limits of complicated functions from limits of simpler functions.

Some simple limits

Proposition. Let c and k be real numbers. Then

1. $\lim_{x \to c} k = k$.

Some simple limits

Proposition. Let c and k be real numbers. Then

1. $\lim_{x \to c} k = k$.
2. $\lim_{x \to c} x = c$.

Some simple limits

Proposition. Let c and k be real numbers. Then

1. $\lim_{x\to c} k = k$.
2. $\lim_{x\to c} x = c$.

Examples.

$$\lim_{x\to 5} 13 = 13$$

Some simple limits

Proposition. Let c and k be real numbers. Then

1. $\lim_{x\to c} k = k$.
2. $\lim_{x\to c} x = c$.

Examples.

$$\lim_{x\to 5} 13 = 13$$

$$\lim_{x\to 5} x = 5.$$

Example of the Limit Theorem in action

Claim. $\lim_{x \to 5} x^2 = 25$.

Example of the Limit Theorem in action

Claim. $\lim_{x \to 5} x^2 = 25.$

Proof.

It's easy to show $\lim_{x \to 5} x = 5$ (and we'll do this next time.)

Example of the Limit Theorem in action

Claim. $\lim_{x\to 5} x^2 = 25$.

Proof.

It's easy to show $\lim_{x\to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the Limit Theorem with $f(x) = g(x) = x$, we get

$$\lim_{x\to 5} x^2$$

Example of the Limit Theorem in action

Claim. $\lim_{x \to 5} x^2 = 25$.

Proof.

It's easy to show $\lim_{x \to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the Limit Theorem with $f(x) = g(x) = x$, we get

$$\lim_{x \to 5} x^2 = \lim_{x \to 5} (x \cdot x)$$

Example of the Limit Theorem in action

Claim. $\lim_{x \to 5} x^2 = 25$.

Proof.

It's easy to show $\lim_{x \to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the Limit Theorem with $f(x) = g(x) = x$, we get

$$\lim_{x \to 5} x^2 = \lim_{x \to 5} (x \cdot x) = \left(\lim_{x \to 5} x \right) \left(\lim_{x \to 5} x \right)$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 5} x^2 = 25$.

Proof.

It's easy to show $\lim_{x\to 5} x = 5$ (and we'll do this next time.) Then, using part 2 of the Limit Theorem with $f(x) = g(x) = x$, we get

$$\lim_{x\to 5} x^2 = \lim_{x\to 5}(x \cdot x) = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5 \cdot 5 = 25.$$

$\square$

Example of the Limit Theorem in action

Claim $\star$. If $\lim_{x \to c} f(x)$ exists and $k \in \mathbb{R}$, then

$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x).$$

Example of the Limit Theorem in action

Claim. $\lim_{x \to 1} \frac{3x^2-5}{x^3-2x+3} = -1$.

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1.$

Proof. Apply the limit theorem:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} =$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1$.

Proof. Apply the limit theorem:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} \qquad \text{(part 3)}$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1.$

Proof. Apply the limit theorem:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} \qquad \text{(part 3)}$$

$$= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x^3 + \lim_{x\to 1} -2x + \lim_{x\to 1} 3)} \qquad \text{(part 1)}$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1.$

Proof. Apply the limit theorem:

$$\begin{aligned}
\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} &= \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} && \text{(part 3)} \\
&= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x^3 + \lim_{x\to 1} -2x + \lim_{x\to 1} 3)} && \text{(part 1)} \\
&= \frac{3(\lim_{x\to 1} x^2) - 5}{\lim_{x\to 1} x^3 - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} && (\star \text{ and Prop.})
\end{aligned}$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1.$

Proof. Apply the limit theorem:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} \qquad \text{(part 3)}$$

$$= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x^3 + \lim_{x\to 1} -2x + \lim_{x\to 1} 3)} \qquad \text{(part 1)}$$

$$= \frac{3(\lim_{x\to 1} x^2) - 5}{\lim_{x\to 1} x^3 - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad (\star \text{ and Prop.})$$

$$= \frac{3(\lim_{x\to 1} x \lim_{x\to 1} x) - 5}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad \text{(part 2)}$$

Example of the Limit Theorem in action

Claim. $\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = -1$.

Proof. Apply the limit theorem:

$$\lim_{x\to 1} \frac{3x^2-5}{x^3-2x+3} = \frac{\lim_{x\to 1}(3x^2-5)}{\lim_{x\to 1}(x^3-2x+3)} \qquad \text{(part 3)}$$

$$= \frac{\lim_{x\to 1} 3x^2 + \lim_{x\to 1}(-5)}{\lim_{x\to 1} x^3 + \lim_{x\to 1} -2x + \lim_{x\to 1} 3)} \qquad \text{(part 1)}$$

$$= \frac{3(\lim_{x\to 1} x^2) - 5}{\lim_{x\to 1} x^3 - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad (\star \text{ and Prop.})$$

$$= \frac{3(\lim_{x\to 1} x \lim_{x\to 1} x) - 5}{\lim_{x\to 1} x \lim_{x\to 1} x \lim_{x\to 1} x - 2\lim_{x\to 1} x + \lim_{x\to 1} 3} \qquad \text{(part 2)}$$

$$= \frac{3(1\cdot 1) - 5}{1\cdot 1\cdot 1 - 2\cdot 1 + 3} = \frac{-2}{2} = -1 \qquad \text{(Prop.)}$$