Math 111

September 2, 2022

Limits

Definition. Let f be a function defined in an open interval containing a point c, except f might not be defined at the point c, itself. Let L be a real number. The *limit of* $f(x)$ *as* x *approaches* c is L, denoted $\lim_{x \to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such if x satisfies

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The definition of the limit contains a huge amount of information. Unless you have worked with it before—which I am not assuming—don't expect to understand it on first reading (or on the second or third, for that matter).

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Example. What if $f(x) = \frac{x^2 - x}{x}$ and $c = 0$?

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So $|f(x) - L| < \varepsilon$ means the distance between $f(x)$ and the number L is less than ε.

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$|x - c| < \delta$ means the distance between x and c is less than δ.

$0 < |x - c|$ means that $x \neq c$.

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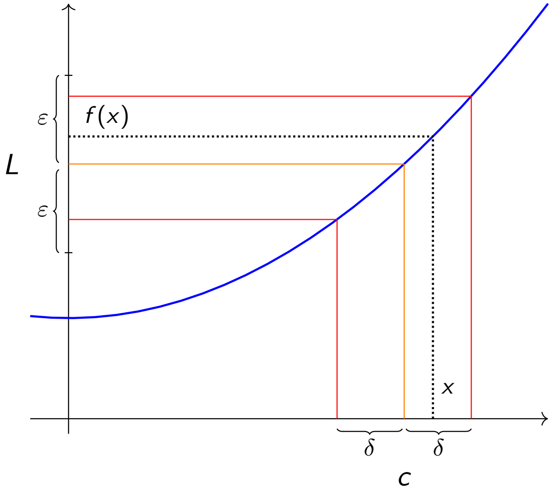
ε is the challenge, and δ is the response:

Given a small distance ε, can you constrain x close enough to c to make $f(x)$ within ε of L?

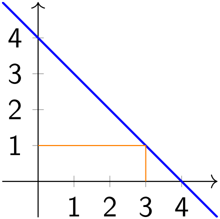
Relevant diagram



$$\lim_{x \to c} f(x) = L$$

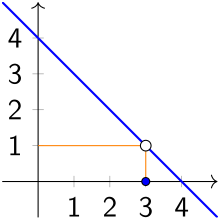
Examples

$\lim_{x \to 3} 4 - x = 1.$



Examples

$$f(x) = \begin{cases} 4 - x & \text{if } x \neq 3 \\ 0 & \text{if } x = 3. \end{cases}$$



Here, $\lim_{x \to 3} f(x) = 1$, again. The limit would be the same even if f were not defined at all at $x = 3$.

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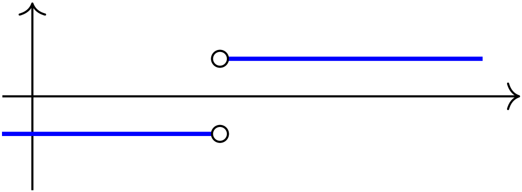

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Given $\varepsilon > 0$, let $\delta = \varepsilon/2$. Suppose that $0 < |x - 3| < \delta$; in other words, suppose that $0 < |x - 3| < \varepsilon/2$. Then

$$|(2x + 5) - 11|$$

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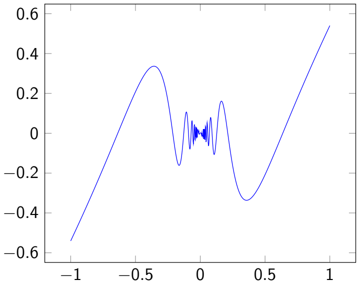

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