01° On $\mathbb{R}^3$, we have the conventional pdo geometry, defined by the following bilinear form:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x$ and $y$ are any members of $\mathbb{R}^3$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Convert the basis:

$$B_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for $\mathbb{R}^3$ to an orthonormal basis, causing minimal disturbance.

02° Let $V$ be the linear space consisting of all polynomials $h$ with real coefficients, having degree no greater than 3:

$$h(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Let $V$ be supplied with a pdo geometry, as follows:

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

where $f$ and $g$ are any polynomials in $V$. Introduce the following basis:

$$\mathcal{B} : \quad b_0, b_1, b_2, b_3$$

for $V$, where:

$$b_j(x) = x^j \quad (0 \leq j \leq 3, \ x \in \mathbb{R})$$

Convert $\mathcal{B}$ to an orthonormal basis for $V$, causing minimal disturbance. Now let $S$ be the linear mapping in $L(V)$, defined by differentiation:

$$S(h) = h'$$

where $h$ is any polynomial in $V$. Describe the adjoint $T$ of $S$. Is $S$ self adjoint?
Let $V'$ and $V''$ be pdo geometries. Let $S$ and $T$ be a linear mappings in $L(V', V'')$ and $L(V'', V')$, respectively. Show that if $S$ and $T$ are adjoints of one another then the compositions $TS$ and $ST$ in $L(V')$ and $L(V'')$, respectively, are self adjoint.

Let $V$ be a pdo geometry. Let $P$ be a linear mapping in $L(V)$ for which $PP = P$. Show that the conditions:

1. $V = \text{ran}(P) \perp \text{ker}(P)$
2. $P$ is self adjoint

are equivalent. Under the condition $PP = P$, we say that $P$ is a projection. Sometimes, we say “orthogonal projection” rather than “self adjoint projection.” Take special note of condition (1). It figures in both the Spectral Theorem and the Singular Value Decomposition. Verify that if $P$ is a self adjoint projection then $Q = I - P$ is also a self adjoint projection, while:

$$\text{ran}(Q) = \text{ker}(P), \quad \text{ker}(Q) = \text{ran}(P)$$

Let us note first that, for any $Y$ in $\text{ran}(P)$, there is some $X$ in $V$ for which $Y = P(X)$, so that $P(Y) = P(P(X)) = P(X) = Y$. Now let us prove that (2) implies (1). To that end, we define $Q = I - P$. Clearly, $Q$ is self adjoint, $P + Q = I$, $PQ = 0 = QP$, and $QQ = Q$. Hence, for any $X$ in $V$, we find that $X = I(X) = P(X) + Q(X)$. Obviously, $P(X)$ lies in $\text{ran}(P)$. Moreover, $Q(X)$ lies in $\text{ker}(P)$, since $PQ = 0$, and so, in turn, $\langle P(X), Q(X) \rangle = \langle X, P(Q(X)) \rangle = 0$. Finally, for any $Y$ in $\text{ran}(P)$ and $Z$ in $\text{ker}(P)$, if $Y + Z = 0$ then $\langle Y, Z \rangle = \langle P(Y), Z \rangle = \langle Y, P(Z) \rangle = 0$. It follows that $0 = \langle Y, Y + Z \rangle = \langle Y, Y \rangle$, so that $Y = 0$, hence that $Z = 0$. We infer that $V = \text{ran}(P) \perp \text{ker}(P)$. In turn, let us prove that (1) implies (2). To that end, let $Y_1$ and $Y_2$ be any members of $\text{ran}(P)$ and let $Z_1$ and $Z_2$ be any members of $\text{ker}(P)$. We obtain:

$$\langle P(Y_1 + Z_1), Y_2 + Z_2 \rangle = \langle Y_1, Y_2 \rangle = \langle Y_1 + Z_1, P(Y_2 + Z_2) \rangle$$

We infer that $S$ is self adjoint. The proof is complete.