Turaev-Viro TQFTs via Planar Algebras

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How To Construct a TQFT:

1. Algebraic Input
2. Combinatorial Description of Manifolds
3. Shake them together...
How To Construct a TQFT:

1. Algebraic Input
2. Combinatorial Description of Manifolds
3. Shake them together...

1. Spherical Tensor Category/Planar Algebra
2. Triangulations of 3-Manifolds
3. The Turaev-Viro Construction...
Theorem (Pachner)

Any two triangulations of $d$-manifolds can be related by a finite sequence of ‘Pachner moves’

$$(k \leftrightarrow d + 2 - k) \quad 1 \leq k \leq d/2 + 1$$
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3D Pachner Moves:

The 2-3 Move:

The 1-4 Move:
Basic Idea: $M = 3$-manifold, $C = \text{Spherical Category}$.  

- triangulate $M^3$, order the vertices.
The Turaev-Viro Construction

Basic Idea: \( M = \) 3-manifold, \( C = \) Spherical Category.

- triangulate \( M^3 \), order the vertices.
- Label triangulation with Data from \( C \)
- Get a number for each tetrahedron.

Multiply the numbers for each tetrahedron.
Sum over labelings.

⇒ Get a Manifold Invariant!

Miracle: This invariant is not trivial!

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The Turaev-Viro Construction

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- (Prove invariance under 2-3 moves, 1-4 moves, and reordering)
  \( \Rightarrow \) Get a Manifold Invariant!
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Properties of Planar Algebras

- Simple Objects \((X, Y, Z, \cdots)\)
- Morphisms

\[
\begin{align*}
Z \\
\sigma \\
X \\
Y
\end{align*}
\]
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- Simple Objects \((X, Y, Z, \cdots)\)
- Morphisms
- Dual Morphisms
- Hermitian Inner Product (positive definite!)

\[ \langle \mu, \sigma \rangle \]

A Basis \(\{\sigma_i\}\) for \(\text{Hom}(X \otimes Y, Z)\) also gives a dual basis \(\{\sigma_i^*\}\) for \(\text{Hom}(Z, X \otimes Y)\).
Basis for $\text{Hom}(X \otimes Y, Z)$ also gives basis for

- $\text{Hom}(Y \otimes Z, X)$ (rotate in planar algebra!)
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- $\text{Hom}(Y \otimes Z, X)$ (rotate in planar algebra!)
- and $\text{Hom}(Z \otimes Y, X)$
Is it orthonormal?

\[ \sigma^* \]

\[ \sigma \]

\[ X \]

\[ Z \]

\[ \bar{Y} \]

\[ Y \]

\[ X \]
Is it orthonormal?

\[ \sigma^* \sigma \quad Z \quad X \quad Y \]

\[ \sigma \quad \overline{Z} \quad \overline{Y} \quad X \]

\[ \cdot d(Z)d(X)^{-1} \]
Is it orthonormal?

\[ \sigma^* \cdot d(Z)d(X)^{-1} \]
Is it orthonormal?

\[ \sigma^* \sigma \cdot d(Z)d(X)^{-1} \]
Is it orthonormal?

\[ \sigma^* \circ \sigma \cdot d(Z)d(X)^{-1} \]

Need to rescale by a factor of \( \sqrt{d(Z)^{-1}d(X)} \).
How to Label a Tetrahedron and get a Number

- Choose representative simples $A, B, C, X, Y, Z, \ldots$
- Choose basis $\{\sigma_i\}$ for each $\text{Hom}(A \otimes B, C)$.
- Label edges with simple objects.
- Label faces with elements $\sigma_i$.
- Then...
Get a formula...

\[ Z(M, \tau) = \sum_{\text{edge labels}} \sum_{\text{face labels}} \prod_{T} Z(T) \]

If orientation of \( T \) disagrees with \( M \), use \( \overline{Z(T)} \).

How to prove invariance?
Is it even invariant?
Reordering Vertices

What happens when we reorder the vertices?

![Diagram showing reordering of vertices]

Correction factor: \[ \sqrt{d(B) - 1} \]
Reordering Vertices

What happens when we reorder the vertices?
Original (13)-edge label: $B$, (02)-edge label: $C$

$Z(T_{0\leftrightarrow 1})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(A)d(D)}$

(13)-edge: $D$, (02)-edge: $A$
What happens when we reorder the vertices?

Original (13)-edge label: $B$, (02)-edge label: $C$

- $Z(T_{0\leftrightarrow 1})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(A)d(D)}$
  (13)-edge: $D$, (02)-edge: $A$

- $Z(T_{1\leftrightarrow 2})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(X)d(Y)}$
  (13)-edge: $X$, (02)-edge: $Y$
Reordering Vertices

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- (13)-edge: $X$, (02)-edge: $Y$

- $Z(T_{1\leftrightarrow 3})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(C)d(B)}$
  
- (13)-edge: $B$, (02)-edge: $C$
Reordering Vertices

What happens when we reorder the vertices?
Original (13)-edge label: $B$, (02)-edge label: $C$

- $Z(T_{0\leftrightarrow 1})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(A)d(D)}$
  (13)-edge: $D$, (02)-edge: $A$
- $Z(T_{1\leftrightarrow 2})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(X)d(Y)}$
  (13)-edge: $X$, (02)-edge: $Y$
- $Z(T_{1\leftrightarrow 3})/Z(T) = \sqrt{d(B)^{-1}d(C)^{-1}d(C)d(B)}$
  (13)-edge: $B$, (02)-edge: $C$

Correction factor: $\sqrt{d(B)^{-1}d(C)^{-1}}$
\[ Z(M, \tau) = \sum_{\text{edge labels}} \sum_{\text{face labels}} \prod_{\text{tetrahedra } T} Z_{\text{corrected}}(T) \]

Invariance:

- Vertex order \( \checkmark \)
- 2-3 Pachner Move
- 1-4 Pachner Move
First Identity:
Second Identity:

\[ \sum_{\sigma, G} \sigma G \]

\[ \lambda \]

\[ \mu \]

\[ \sigma^* \]

\[ \sigma \]

\[ X \rightarrow Y \]

\[ Y \rightarrow X \]

\[ A \rightarrow B \]
2-3 Move
Second Half of 2-3 Move
What about the correction factors?

- Correction factor of first half (2)
  \[ \sqrt{d(B)^{-1}d(C)^{-1}} \sqrt{d(H)^{-1}d(Y)^{-1}} \]

- Correction factor of second half (3)
  \[ \sqrt{d(B)^{-1}d(G)^{-1}} \sqrt{d(H)^{-1}d(C)^{-1}} \sqrt{d(G)^{-1}d(Y)^{-1}} \]
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- Correction factor of second half (3)
  \[ \sqrt{d(B)^{-1}d(G)^{-1}} \sqrt{d(H)^{-1}d(C)^{-1}} \sqrt{d(G)^{-1}d(Y)^{-1}} \]

They differ by \( d(G)^{-1} \)
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- Correction factor of second half (3)
  \[ \sqrt{d(B)^{-1}d(G)^{-1}} \sqrt{d(H)^{-1}d(C)^{-1}} \sqrt{d(G)^{-1}d(Y)^{-1}} \]

They differ by \( d(G)^{-1} \)

Edge Correction Factor: \( \prod_{\text{edges}} d(\text{edge label}) \)
\[ Z(M, \tau) = \sum_{\text{edge labels}} \sum_{\text{face labels}} \prod_{\text{tetrahedra} T} Z_{\text{cor}}(T) \prod_{\text{edges}} \text{(edge factor)} \]

Invariance:

- Vertex order \( \checkmark \)
- 2-3 Pachner Move \( \checkmark \)
- 1-4 Pachner Move
What about the 1-4 Move?

Exercise

Prove that the two sides of the 1-4 Move differ by a factor of $w = \sum E d(E)^2$ where $E$ ranges over a complete set of simples.
What about the 1-4 Move?

Exercise

Prove that the two sides of the 1-4 Move differ by a factor of

\[ w = \sum_{E} d(E)^2 \]

where \( E \) ranges over a complete set of simples.
The Turaev-Viro Invariant

\[ Z(M, \tau_{\text{triangulation}}) = \frac{1}{\# \text{vertices}} \cdot \sum_{\text{edge labels}} \sum_{\text{face labels}} \prod_{\text{tetrahedra } T} Z_{\text{cor}}(T) \prod_{\text{edges}} (\text{edge factor}) \]

Invariance:
- Vertex order ✓
- 2-3 Pachner Move ✓
- 1-4 Pachner Move ✓
Exercise

For the $A_2$ planar algebra, for $M^3$ closed,

$$Z(M) = \#(H^1(M; \mathbb{Z}/2)).$$