Algebraic Deformations of Rational Functions

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February 21, 2013
Outline:

Algebraic puzzle

Linking rational functions w/polynomials
Outline:

Algebraic puzzle → Topological avatar

Linking rational functions w/polynomials → Wrapping spheres around spheres
Outline:

- Algebraic puzzle
  - Linking rational functions w/polynomials
- Quadratic solution
  - Bilinear forms up to isometry
- Topological avatar
  - Wrapping spheres around spheres
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Algebraic puzzle

Topological avatar

Linking rational functions w/polynomials

Wrapping spheres around spheres

Quadratic solution

Motivic homotopy

Bilinear forms up to isometry

Where the wild things are
Outline:

- Algebraic puzzle
- Topological avatar

Linking rational functions w/polynomials

- Quadratic solution
- Motivic homotopy

Wrapping spheres around spheres

Bilinear form up to isometry

Where the wild things are.
A puzzle:

Rational functions

\[ f = \frac{p}{q} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots + b_0}, \]

\[ g = \frac{p'}{q'} = \frac{x^n + a'_{n-1}x^{n-1} + \cdots + a'_0}{b'_{n-1}x^{n-1} + \cdots + b'_0}. \]
A puzzle:

Rational functions

\[ f = \frac{p}{q} = \frac{X^n + a_{n-1}X^{n-1} + \cdots + a_0}{b_{n-1}X^{n-1} + \cdots + b_0} \]

\[ g = \frac{p'}{q'} = \frac{X^n + a_{n-1}'X^{n-1} + \cdots + a_0'}{b_{n-1}'X^{n-1} + \cdots + b_0'} \]

E.g.

\[ f = \frac{X^4 - X^2}{-X^2 + 4} \]

\[ g = \frac{X^4 + X^2}{-X^2 - 4} \]
A puzzle:

Rational functions

\[ f = \frac{\varphi}{\varrho} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots + b_0}, \]

\[ g = \frac{\varphi'}{\varrho'} = \frac{x^n + a'_{n-1}x^{n-1} + \cdots + a'_0}{b'_{n-1}x^{n-1} + \cdots + b'_0}. \]

When is there a new rational function

\[ H(X,T) = \frac{P}{Q} = \frac{X^n + A_{n-1}(T)x^{n-1} + \cdots + A_0(T)}{B_{n-1}(T)x^{n-1} + \cdots + B_0(T)}, \]

such that

\[ H(X,0) = f, \quad H(X,1) = g? \]

[The \( A_i(T) \) and \( B_i(T) \) are polynomials.]
Refining the puzzle:

Assume all coefficients are real numbers. For a rational function $H(X,T)$ and complex number $t$, let $H_t = H(X,t) = \frac{P_t}{Q_t}$. 
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If $H_0 = f$ and $H_t = g$, call $H$ an algebraic deformation of $f$ to $g$ and write $H : f \sim g$. 
Example:

\[
\frac{x^4 - x^2}{-x^2 + 4} \quad \frac{x^4 + x^2}{-x^2 - 4}
\]

\[
H(x, T) = \frac{x^4 + (2T - 1)x^2 - 48T^2 + 48T}{-x^2 - 8T + 4}
\]

http://math.mit.edu/~ormsby/alg_def.gif
Non-example:

\[ H(X,T) = \frac{X^2}{X + 2T - 1} \]

We have

\[ H(X,0) = \frac{X^2}{X - 1} \]
\[ H(X,1) = \frac{X^2}{X + 1} \]

but...

http://math.mit.edu/~ormsby/not_def.gif
The final puzzle:

Classify rational functions

\[ f = \frac{p}{q} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_{n-1}x^{n-1} + \cdots + b_0} \]

up to **ALGEBRAIC EQUIVALENCE**.
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up to ALGEBRAIC EQUVALENCE.

Two degree \( n \) rational functions \( f, g \) are algebraically equivalent if there is a chain of algebraic deformations

\[ f = f_0 \stackrel{H_1}{\sim} f_1 \stackrel{H_2}{\sim} f_2 \stackrel{H_m}{\sim} \ldots \stackrel{H_m}{\sim} f_m = g \]
Maps of spheres:

Plugging real or complex values into $f = \frac{p}{q}$,

$f_R : \mathbb{R} \longrightarrow \mathbb{R}$

$f_c : \mathbb{C} \longrightarrow \mathbb{C}$

NOT defined when $q(x) = 0$. 
Maps of spheres:

Plugging real or complex values into $f = \frac{p}{q}$,

$\begin{align*}
    f_{\mathbb{R}} &: \mathbb{R} \longrightarrow \mathbb{R} \\
    f_{\mathbb{C}} &: \mathbb{C} \longrightarrow \mathbb{C}
\end{align*}$

NOT defined when $q(x) = 0$.

So add a POINT AT INFINITY:

$\begin{align*}
    f_{\mathbb{R}} &: S^1 \longrightarrow S^1
\end{align*}$
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So add a POINT AT INFINITY:

$\infty$ $\mathbb{S}^2$

$f_c : \mathbb{S}^2 \rightarrow \mathbb{S}^2$
Maps of spheres (ct'd):

So real rational functions produce continuous maps

\[ f_R : S^1 \to S^1 \] \quad \text{If } q(x) = 0, \text{ then } f(x) = \infty;
\]

\[ f_c : S^2 \to S^2 \] \quad \text{also } f(\infty) = \infty.\]
Maps of spheres (ct'd):

So real rational functions produce continuous maps
\[ f_R : S^1 \rightarrow S^1 \]  If \( q(x) = 0 \), then \( f(x) = \infty \);
\[ f_C : S^2 \rightarrow S^2 \]  also \( f(\infty) = \infty \).

If we let \( T \) take values in the interval \( I = [0,1] \),
then \( H : f \simeq g \) induces HOMOTOPY
\[ H_R : f_R \simeq g_R, \quad H_C : f_C \simeq g_C. \]
Maps of spheres (ct'd):

So real rational functions produce continuous maps
\[ f_R : S^1 \to S^1 \]  ? If \( q(x) = 0 \), then \( f(x) = \infty \);
\[ f_c : S^2 \to S^2 \]  also \( f(\infty) = \infty \).

If we let \( T \) take values in the interval \( I = [0,1] \),
then \( H : f \sim g \) induces HOMOTOPIES
\[ H_R : f_R \sim g_R, \quad H_c : f_c \sim g_c. \]

On "real points" we can view \( H_R \) as

\[ I \quad \xrightarrow{H_R} \quad S^1 \]

"continuously varying family of maps \( S^1 \to S^1 \)."
A topological variation:

On "complex points" we get...

On the diagram, the text reads:

$$H_C$$

"continuous family of maps $S^2 \rightarrow S^2$"
A topological variation:

On "complex points" we get:

\[ \alpha, \beta : X \to Y \text{ are homotopic if there is a homotopy } H : X \times \mathbb{I} \to Y \text{ such that } H_0 = \alpha, H_1 = \beta. \]

An algebraic deformation induces homotopies:

\[ H_R : f_R \simeq g_R, \quad H_c : f_c \simeq g_c \]
Wrapping spheres around spheres:

Let's address the simpler problem of classifying continuous maps $S^1 \to S^1$ and $S^2 \to S^2$ (that send $\infty$ to $\infty$) up to homotopy.
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**EXAMPLES:**

[Diagram of spheres with marked points]
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**Examples:**

[pinch diagram]
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**EXAMPLES:**

![Diagram showing the concept of wrapping spheres around spheres with labels for pinch and fold.](image)
The degree of a map:

We may assume $f : S^n \rightarrow S^n$ is smooth and choose a regular value $y$ in its target.
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If $f^{-1}(y) = \{x \in S^n \mid f(x) = y\}$, then $|f^{-1}(y)|$ changes by an even number under homotopy. The value $|f^{-1}(y)| \mod 2$ is called the MOD 2 DEGREE of $f$. 
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Can we do better?
The degree of a map:

Label each point in $f^{-1}(y)$ with a sign:

+1 if $f$ locally PRESERVES ORIENTATION
-1 if $f$ locally REVERSES ORIENTATION
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Label each point in $f^{-1}(y)$ with a sign:

$+1$ if $f$ locally **PRESERVES ORIENTATION**

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$$
\begin{align*}
\text{degree 1} & \quad \begin{cases} 
+1 \\
-1 \\
+1
\end{cases} \\
\end{align*}
$$
The degree of a map:

Label each point in \( f^{-1}(y) \) with a sign:

+1 if \( f \) locally PRESERVES ORIENTATION
-1 if \( f \) locally REVERSES ORIENTATION

Homotopies only cancel points with opposite sign, so the sum of the signs in \( f^{-1}(y) \) is invariant under homotopy: the DEGREE of \( f \), \( \text{deg}(f) \).
Brouwer's Theorem:

The degree map is a bijective correspondence between homotopy classes of maps $S^n \to S^n$ and the integers, $\mathbb{Z}$, for $n \geq 1$. 
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The degree map is a bijective correspondence between homotopy classes of maps $\mathbb{S}^n \to \mathbb{S}^n$ and the integers, $\mathbb{Z}$, for $n \geq 1$.

Consequence: If $f$ and $g$ are algebraically equivalent rational functions, then

$$\deg(f_R) = \deg(g_R) \quad \text{and} \quad \deg(f_C) = \deg(g_C)$$

Question: What algebraic data determines $\deg(f_R)$ and $\deg(f_C)$?
**Degrees of rational functions:**

**Proposition:** If \( f = \frac{P}{Q} = \frac{x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_{n-1}x^{n-1} + \ldots + b_0} \) is a degree \( n \) rational function, then \( \deg(f_c) = n \).
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**Proof sketch:** Consider \( f_c^{-1}(0) \). We have

\[
|f_c^{-1}(0)| = |P^{-1}(0)| = |\{ \text{roots of } P \}|
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If \( 0 \) is a regular value of \( f_c \) then this number is \( n \).
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If \( 0 \) is a regular value of \( f_c \) then this number is \( n \).

**Exercise:** \( f_c \) preserves orientation everywhere.
Proposition: If $f'(x) \neq 0, \deg(f_R) = \sum \frac{\text{sign}(f'(x))}{f(x) = 0}$.
Degrees of rational functions:

\[ \deg(f_R) = 1 + 1 - 1 + 1 = 2 \]

**Proposition:** If \( f'(x) \neq 0 \) and \( f(x) = 0 \), then

\[ \deg(f_R) = \sum_{f(x) = 0} \text{sign}(f'(x)). \]
A relationship between degrees:

If $f$ is a degree $n$ rational function, we know that $\deg(f_c) = n$.

How does $\deg(f_{\mathbb{R}})$ compare to $\deg(f_c)$?
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How does \( \deg(f_{R}) \) compare to \( \deg(f_c) \)?

Observe:

- \( f \) has at most \( n \) real roots
- non-real roots come in conjugate pairs
- \( \deg(f_{R}) = n - (\# \text{ non-real roots}) - 2 (\# \text{ roots } x \text{ with } f'(x) < 0) \)
A relationship between degrees:

If $f$ is a degree $n$ rational function, we know that $\deg(f_c) = n$.

How does $\deg(f_{IR})$ compare to $\deg(f_c)$?

Observe:
- $f$ has at most $n$ real roots
- non-real roots come in conjugate pairs
- $\deg(f_{IR}) = n - (\#\text{non-real roots}) - 2(\#\text{roots w/ } f'(x) < 0)$

Proposition: $-n \leq \deg(f_{IR}) \leq n$ and $\deg(f_{IR}) = n$ (2).
Summary:

If $f$ and $g$ are algebraically equivalent, we know that $\deg(f_R) = \deg(g_R)$ and $\deg(f_C) = \deg(g_C)$.

We also know $n = \deg(f_C)$ is a non-negative integer and $m = \deg(f_R)$ satisfies $-n \leq m \leq n$, $m \equiv n \pmod{2}$. 
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We also know $n = \deg(f_C)$ is a non-negative integer and $m = \deg(f_R)$ satisfies $-n \leq m \leq n$, $m = n$ (2).

Are all such pairs $(m, n)$ realized as $(\deg(f_R), \deg(f_C))$ for some $f$?
Summary:

If $f$ and $g$ are algebraically equivalent, we know that $\deg(f_{\mathbb{R}}) = \deg(g_{\mathbb{R}})$ and $\deg(f_{\mathbb{C}}) = \deg(g_{\mathbb{C}})$.

We also know $n = \deg(f_{\mathbb{C}})$ is a non-negative integer and $m = \deg(f_{\mathbb{R}})$ satisfies $-n \leq m \leq n$, $m = n$ (2).

Q1 Are all such pairs $(m, n)$ realized as $(\deg(f_{\mathbb{R}}), \deg(f_{\mathbb{C}}))$ for some $f$?

Q2 Does $(\deg(f_{\mathbb{R}}), \deg(f_{\mathbb{C}})) = (\deg(g_{\mathbb{R}}), \deg(g_{\mathbb{C}}))$ imply $f$ is algebraically equivalent to $g$?
The resultant:

Polynomials \( p = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \), 
\( q = b_{n-1}x^{n-1} + \cdots + b_0 \)

have no common root if and only if their RESULTANT is a unit.
The resultant:

Polynomials \( p = x^n + a_{n-1} x^{n-1} + \ldots + a_0, \)
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have no common root if and only if their \underline{RESULTANT} is a unit.

E.g. If \( p = x^3 + 5x^2 + 7x - 3 \) and \( q = 2x^2 - x + 11 \), then

\[
\text{res}(p, q) = \det \begin{pmatrix}
1 & 2 \\
5 & -1 & 2 \\
7 & 5 & 11 & -1 & 2 \\
-3 & 7 & 11 & -1 & 2 \\
-3 & 11 & \end{pmatrix}
\]

\( \in \mathbb{Z} \) Sylvester matrix

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The resultant:

If \( H = \frac{P}{Q} : f \sim g \) then we can form \( \text{res}(P, Q) \) as a polynomial in \( T \).

In order for \( H \) to be an algebraic deformation, \( \text{res}(P, Q) \) must be constant and nonzero.
The resultant:

If $H = \frac{P}{Q} : f \approx g$ then we can form $\text{res}(P, Q)$ as a polynomial in $T$.

In order for $H$ to be an algebraic deformation, $\text{res}(P, Q)$ must be constant and nonzero.

So if $f = \frac{p}{q}$, $g = \frac{p'}{q'}$, then

$$\text{res}(p, q) = \text{res}(P, Q) = \text{res}(p', q').$$

Let $\text{res}(f) = \text{res}(p, q)$. 
The solution:

Notation $\mathcal{F} = $ pointed rational functions

$\pi_0 \mathcal{F} = $ pt’d rat’l f’ns up to algebraic equivalence

$\pi_0 \mathcal{F} \cong \{(n, m, \lambda) \mid n \in \mathbb{N}, m \in \mathbb{Z}, \lambda \in \mathbb{R}, -n \leq m \leq n, n \equiv m \ (2), (-1)^{(n^2 - m)/2} \lambda > 0\}$

$[f] \mapsto (\deg(f_{\mathbb{C}}), \deg(f_{\mathbb{R}}), \text{res}(f))$
A generalization:

What if the coefficients of our functions were in:
- $\mathbb{C}$
- $\mathbb{Q}$
- $\mathbb{Z}/p\mathbb{Z}$
- some other field

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Could we still classify rational functions up to algebraic equivalence?
A generalization:

What if the coefficients of our functions were in:
- \( \mathbb{C} \) — just use \( \deg(f_c) \) and \( \text{res}(f) \)
- \( \mathbb{Q} \) — current methods give partial results
- \( \mathbb{Z}/p\mathbb{Z} \) — subtle
- some other field — a uniform answer?

Could we still classify rational functions up to algebraic equivalence?
The Bezout form:

For polynomials $p(X), q(X)$, we have

$$X-Y \mid p(x)q(y) - p(y)q(x).$$

Hence

$$S_{p,q}(X,Y) = \frac{p(X)q(Y) - p(Y)q(X)}{X-Y}.$$

$$= \sum_{1 \leq k, \ell \leq n} c_{k,\ell} X^{k-1} Y^{\ell-1}.$$
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$$= \sum_{1 \leq k, l \leq n} c_{k,l} x^{k-1} y^{l-1}.$$

The BEZOUT FORM of $f = \frac{p}{q}$ is the bilinear form

$$\text{Bez}(f)(x, y) = \sum_{1 \leq k, l \leq n} c_{k,l} x_k y_l.$$

with matrix $(c_{k,l})$. 
Bezout form as derivative:

The Bezout form of $f$ is closely related to $f'$.

$$\lim_{x \to y} \delta_{p,q}(x,y) = \lim_{x \to y} \frac{p(x)q(y) - p(y)q(x)}{x-y}$$
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$$= \lim_{x \to y} \frac{p(x) - p(y)}{x-y} q(y) - p(y) \frac{q(x) - q(y)}{x-y}$$

$$= p'(y)q(y) - p(y)q'(y)$$
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$$= p'(y)q(y) - p(y)q'(y)$$

$$= f''(y) \cdot q(y)^2.$$
Bezout form as derivative:

The Bezout form of $f$ is closely related to $f'$.

$$\lim_{x \to y} \delta_{p,q}(x,y) = \lim_{x \to y} \frac{p(x)q(y) - p(y)q(x)}{x - y}$$

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$$= p'(y) q(y) - p(y) q'(y)$$

$$= f'(y) \cdot q(y)^2.$$ 

- If we're working over $\mathbb{R}$, this quantity is closely related to $\deg(f_R)$.

- We should think of $\text{Bez}(f) = (c_{k,1})$ as an algebraic replacement for differential data.
A solution over any field:

Two bilinear forms $B, B'$ are \textbf{ISOMETRIC} if there is an invertible matrix $A$ such that $A^tBA = B'$.

Let $\text{Bil}(k) =$ isometry classes of bilinear forms over the field $k$. 
A solution over any field:

Two bilinear forms $B, B'$ are ISOMETRIC if there is an invertible matrix $A$ such that $A^T B A = B'$.

Let $\text{Bil}(k) = \text{isometry classes of bilinear forms over the field } k$.

**Theorem (Cazanave, Morel, Barge, Lannes)**

Algebraic equivalence classes of rat’l fnns are completely determined by $\text{Bez}$ and res. There is precisely one for each isometry class $B$ and scalar $\lambda \in k^*$ such that $(-1)^{(m-1)/2} \lambda \det(B) \in (k^*)^2$. 
Classification examples:

- $\text{Bil}(R) = \mathbb{N} \times \mathbb{N}$
  classified by dimension and signature
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- $\text{Bil}(\mathbb{C})$
  classified by dimension
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- $\text{Bil}(\mathbb{Q})$
  studied via completions — Hasse principle
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- $\text{Bil}(\mathbb{Z}/\mathfrak{q}\mathbb{Z})$ [or $\text{Bil}(\mathbb{F}_q)$]
  only two isometry classes in each dimension
Classification examples:

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  only two isometry classes in each dimension

- $\text{Bil}(\text{general field})$ — long and interesting history
Motivic Homotopy:
Motivic Homotopy:

• Rational functions are really $P_k' \rightarrow P_k$.

• Alg. def’s are really $P_k' \times /A_k' \rightarrow P_k'$ [naïve $A'$-homotopies].

• With enough ALGEBRAIC TOPOLOGY and ALGEBRAIC GEOMETRY applied, we have a brand new tool that answers decades-old questions about bilinear forms.
Thank you!

Slides and animations available at http://math.mit.edu/~ormsby/