Cartesian products, relations, & equivalence relations

Cartesian products keep track of pairs of objects in sets. Unlike sets, they are sensitive to order and permit duplication.

Def: For sets $A, B$, $A \times B = \{ (a, b) \mid a \in A, b \in B \}$

ordered pair

But what is an “ordered pair”? Intuitively, it’s a structure which records one element of $A$ and one element of $B$. In order to minimize the number of primitive objects in our mathematical universe, we seek a way to define such a structure using sets. To this end:

$$(a, b) = \{ \{a\}, \{a, b\}\} .$$

Note: If $a = b$, we have $(a, b) = \{\{a\}, \{a, a\}\} = \{\{a\}\}$ — weird, but OK!

Prop: For $a, c \in A$, $b, d \in B$, the (set-theoretically defined) $(a, b)$ and $(c, d)$ are equal iff $a = b$ and $c = d$.

Reading: 

How do we see $A \times B$?

$R \times R$ is the plane familiar from geometry.

Notation $A \times A = A^2$, $A \times A \times \cdots \times A = A^n$ -- $n$-fold Cartesian product
Subsets of Cartesian products play a special role. We can use them to define familiar (\(=, <, \geq, \)), divisible by, \(\cdots\)) and unfamiliar relations.

**Defn.** Any subset of \(A \times B\) is called a relation on \(A\) and \(B\).

A relation on \(A\) is defined to be a relation on \(A\) and \(A\) (so a subset of \(A \times A\)). If \(R \subseteq A \times B\) is a relation on \(A\) and \(B\), we write \((a, b) \in R\).

**e.g.**

1. \(\leq\) is a relation on \(\mathbb{R}\) consisting of pts on or above the line \(y = x\) in \(\mathbb{R} \times \mathbb{R} = \mathbb{R}^2\).

2. \(=\) is a relation on any set \(A\) consisting of ordered pairs \((a, a), a \in A\).

3. Relations are ridiculously general. There are in fact \(64\) different relations on \(A = \{1, 2\}\) and \(B = \{a, b, c\}\). (Check this after doing Exercise 2.2.1.)

Let's impose some restrictions on our relations so that they are more meaningful:

**Defn.** Let \(R\) be a relation on \(A\).

1. \(R\) is reflexive if \(\forall a \in A, aRa\).
2. \(R\) is symmetric if \(\forall a, b \in A, aRb \Rightarrow bRa\).
3. \(R\) is transitive if \(\forall a, b, c \in A, (aRb \land bRc) \Rightarrow aRc\).
4. \(R\) is an equivalence relation if it is reflexive, symmetric, and transitive.
e.g.,

1. \( \leq \) on \( \mathbb{R} \) is reflexive and transitive, but not symmetric.
2. = on any set \( A \) is an equivalence reln.
3. Let \( A = \{ \text{people} \} \) and \( C \subseteq A \times A \) s.t. \((x,y) \in C \) iff \( x \) is the cousin of \( y \). \( C \) is symmetric but is not reflexive or transitive.

**Defn.** Let \( R \) be an equivalence relation on \( A \). For each \( a \in A \), the set \( [a] = \{ b \in A \mid aRb \} \) is the equivalence class of \( a \).

e.g.

- \( A = \{ \text{students living in campus housing} \} \)
- Define \( R \) so that \( aRb \) precisely when \( a \) and \( b \) live in the same dorm. Then for any \( a \in A \), \([a] = \{ \text{students who live in the same dorm as } a \} \). Is this an equiv reln? Yes.

**Thm.** Let \( R \) be any equiv reln on a set \( A \). Two equiv classes \([a], [b] \subseteq A\) satisfy either \([a] = [b]\) or \([a] \cap [b] = \emptyset\).

**Pf.** Let \( a, b \in A \). If \([a] \cap [b] = \emptyset\), then that matches one of our conclusions and we are done. Thus we may assume \([a] \cap [b] \neq \emptyset\) in which case \( \exists c \in [a] \cap [b] \). Let \( d \) be any elt of \([a]\). Then \( aRd, cRd, \) and \( bRd \). By symmetry, \( dRa \), thus by transitivity \( dRc \). By symmetry, \( cRb \), so again by trans \( dRb \). By symmetry, \( bRd \), so \( d \in [b] \) as well. Thus \([a] \subseteq [b] \).

Swapping the roles of \( a \) and \( b \) above, we see that \([b] \subseteq [a]\) as well, and thus \([a] = [b]\), as desired.

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**Note.** When \( R \) is an equiv reln on \( A \), we get that is the union of disjoint equiv classes. In other words, the equivalence classes partition \( A \). A partition of \( A \) is a set of subsets \( A, \subseteq A \), i.e.
such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A$ is the union of the $A_i$.
For such a partition, $\{A_i\}_{i \in I}$, $R = A \times A$ s.t. $aRb$ iff $j \in I$ s.t. $a,b \in A_j$ defines an equivalence relation on $A$.

\[a \equiv b \] is a positive integer. Define a relation $\equiv$ on $\mathbb{Z}$ given by $a \equiv b \iff a - b$ is a multiple of $n$. This is called congruence modulo $n$.

Check carefully: reflexive, symmetric, transitive

Let $\mathbb{Z}/\mathbb{Z} = \{\text{equiv classes for } \equiv\}$

\[= \{[0], [1], [2], \ldots, [n-1]\}\]

Note $[n] = [0], [n+1] = [1]$, etc., so $\mathbb{Z}/\mathbb{Z}$ has at least $n$ elts.

In fact, for $0 \leq i,j < n$, $[i] = [j]$ iff $i \equiv j$, so $\mathbb{Z}/\mathbb{Z}$ has exactly $n$ elts.

We can do arithmetic w/ $\mathbb{Z}/\mathbb{Z}$ exactly as we would on a clock w/ hours $0, 1, 2, \ldots, n-1$.  

If we wish to record which $n$ we are working with, write $a \equiv b \ (\text{mod } n)$.  