Proposition 5: Definition of Sets

\[ \{ x \mid P(x) \} \]

\[ \{ x \in A \mid P(x) \} \]

\[ \{ x \in \mathbb{R} \mid 0 \leq x \} = [0, \infty) \]

\[ \{ x \in \mathbb{R} \mid x^2 = 1 \} = \{-1, 1\} \]

\[ \{ x \in \{ \text{brown} \} \mid x \text{ has brown hair} \} \]

Special sets:

\[ \mathbb{N} = \{ \text{natural} \} = \{ 0, 1, 2, 3, \ldots \} \]

\[ \mathbb{Z} = \{ \text{integers} \} = \{ 0, 1, 2, 3, \ldots, -1, -2, -3, \ldots \} \]

\[ \mathbb{Q} = \{ \text{rational} \} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \} \]

\[ \mathbb{R} = \{ \text{real} \} \]

\[ \mathbb{C} = \{ \text{complex} \} = \mathbb{R} + \sqrt{-1} \]
Relationships & operations w/ sets

Sets A, B. Say \( A \) is a subset of \( B \), written \( A \subseteq B \), when every \( a \in A \) is also an element of \( B \).

\[ \begin{array}{c}
\emptyset \\
\infty \\
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}
\end{array} \]

\( A = B \) when \( A \) & \( B \) contain precisely the same elements: \( A \subseteq B \), \( B \subseteq A \).

\( A \) is a proper subset of \( B \) when \( A \subseteq B \) and \( A \not\subseteq B \). Write \( A \subset B \).

\( \emptyset \subseteq A \), \((0, 1) \subseteq \mathbb{R} \)

\( A \subseteq A \), \( \emptyset \subseteq A \)

\( \subseteq \) is not the same as \( \in \)

**Defn** The intersection of sets \( A, B \) is the collection of objects in both \( A \) and \( B \):

\[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \]
The union of $A$ and $B$ is the set of objects in $A$ or $B$:

$$A \cup B = \{ x | x \in A \text{ or } x \in B \}$$

Reading $\bigcap_{k \in I} A_k$

$\bigcup_{k \in I} A_k$

The complement of $A$ in $B$ is

$$B \setminus A = \{ b \in B | b \notin A \}$$
How do we show sets $A, B$ are equal?

Common technique: Show $A \subseteq B \land B \subseteq A$.

e.g. Let $\{3m+5n \mid m, n \in \mathbb{Z}\} = A$. Then $A = \mathbb{Z}$.

It is easy to see that $A \subseteq \mathbb{Z}$. Next show that $\mathbb{Z} \subseteq A$.

Let $x$ be some integer. By the division algorithm, $x = 3l + r$ where $l, r \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$.

If $r = 0$, then $x = 3 \cdot l + 0 = 3 \cdot l + 5 \cdot 0$ and we can take $m = l$, $n = 0$ to see that $x \in A$. If $r = 1$,

then $x = 3 \cdot l + 1 = 3 \cdot l + (3 \cdot 2 + 5 \cdot (-1))$

$$= 3(l+2) + 5 \cdot (-1) \in A.$$ 

If $r = 2$, then $x = 3 \cdot l + 2 = 3 \cdot l + (3 \cdot (-1) + 5 \cdot (1))$

$$= 3(l-1) + 5 \cdot 1 \in A.$$ 

Thus $x$ is always a member of $A$, whence $\mathbb{Z} \subseteq A$.

Since $A \subseteq \mathbb{Z}$ & $\mathbb{Z} \subseteq A$, we know $A = \mathbb{Z}$. $\square$
Prop: For all sets $A, B, C$, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Aside: Mnemonic: $\emptyset \leftrightarrow \times \leftrightarrow \land$

$\cup \leftrightarrow + \leftrightarrow \lor$

Distribution!