<table>
<thead>
<tr>
<th>Situation</th>
<th>$\forall x. P(x)$</th>
<th>$\exists x. \neg P(x)$</th>
<th>$\exists x. P(x)$</th>
<th>$\forall x. \neg P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No $x$ of specified type</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>There are $x$ of specified type &amp; $P$ is true for all such $x$</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>There are $x$ of specified type &amp; $P$ is false for all such $x$</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>There are $x$ of specified type &amp; $P$ is true for some of these, false for others</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Thus we have proved

Prop: The negation of $\forall x. P(x)$ is $\exists x. \neg P(x)$; the negation of $\exists x. P(x)$ is $\forall x. \neg P(x)$.
What do we prove? And how do we prove it?

$P \lor Q$: Suppose $P = F$, show that $Q = T$. (Or vice versa)

- **e.g.** A pos prime is odd or 2.
- **Pf** Suppose $p$ is a pos prime which is not odd. Then $p = 2^r \cdot r$.
  
  Since $p$ is prime, $r > 1$, so $p = 2$. \(\square\)

$P \Rightarrow Q$: Assume $P = T$, show that $Q = T$.

- **or by \textit{contrapositive}:** Assume $Q = F$, show that $P = F$.

- **e.g.** If an integer $n$ is a mult of 2 \& mult of 3, then $n$ is a mult of 6.

- **Pf** Let $n$ be a mult of 2 \& 3. Then $n = 2p = 3q$, for some integers $p, q$. $2p = 3q$ forces $q$ even so that $q = 2r$ for int $r$.
  
  Thus $n = 3 \cdot 2 \cdot r = 6r$, and we see $n$ is a mult of 6. \(\square\)

**Ex. P(x):** Let $x$ be an arbitrary of specified type. Prove $P(x) = T$.

- **e.g.** All real $\#s$ $x$ satisfy $x^2 = (-x)^2$.

- **Pf** Let $x$ be an arb real $\#$. Then $(-x)^2 = (-x)\cdot (-x) = (1) \cdot x \cdot (-1) \cdot x$ $= (-1) \cdot (-1) \cdot x \cdot x = 1 \cdot x^2 = x^2$, as desired. \(\square\)

**Ex. P(x):** Find or construct $x$ such that $P(x) = T$.

- **or involve a theorem guaranteeing the existence of such $x$.**

- **e.g.** There exists real $\# x$ s.t. $x^3 - 3x = 2$.

- **Pf** Let $x = 2$, and observe that $2^3 - 3 \cdot 2 = 8 - 6 = 2$, as desired. \(\square\)
e.g. There exists real $x$ s.t. $x^2 - x = 1$.

If $f(x) = x^2 - x \in \mathbb{R}$ s.t. $f(0) = 0$, $f(2) = 8$. Since $0 < 1 \leq 2$,

IVT implies $\exists x$. $0 < x < 2$ and $f(x) = 1$.

Then \[ \text{[Euclid, 300 BCE]} \] There are infinitely many prime #s.

If (by contradiction) suppose there are only finitely many prime #s $p_1, p_2, \ldots, p_n$. Let $a = p_1 \cdot p_2 \cdots p_n + 1$. Since $2$ is a prime, $a > 1$.

By FTA, $a$ has a prime factor $p = p_i$ for some $1 \leq i \leq n$.

Since $p$ divides $a$ & $p$ divides $p_1 \cdot p_2 \cdots p_n$, it must divide $a - p_1 \cdot p_2 \cdots p_n = 1$, which is absurd. Thus there are in fact only many primes. \[ \square \]

In order to negate this, we had to negate a statement:

\[\begin{array}{c|c}
\text{Statement} & \text{Negation} \\
\hline p & \neg p \\
\neg (p \land q) & \neg p \lor \neg q \\
\neg (p \lor q) & \neg p \land \neg q \\
\neg (p \Rightarrow q) & p \land \neg q \\
\neg (\forall x. \ P(x)) & \exists x. \neg P(x) \\
\neg (\exists x. \ P(x)) & \forall x. \neg P(x)
\end{array}\]