Topology & density: $\mathbb{R}$ & $\mathbb{C}$

Both $\mathbb{R}$ & $\mathbb{C}$ have an absolute value $|| \cdot ||$ which then gives distance: $|x-y|$ is the distance between $x$ & $y$.

This allows us to make some unified definitions for each of $\mathbb{R}$ & $\mathbb{C}$:

Define $B = \mathbb{R}$ or $\mathbb{C}$. Then the open ball centered at $a \in B$ of radius $r \in \mathbb{R}_{\geq 0}$ is

$$B_B(a, r) = B(a, r) = \{ x \in B \mid |x-a| < r \}$$

So $B_\mathbb{R}(1, \frac{1}{2}) = \{ x \in \mathbb{R} \mid |x-1| < \frac{1}{2} \}$

is the set of all points in $\mathbb{R}$ that are less than $\frac{1}{2}$ units away from $1$.

I.e. $B_\mathbb{R}(1, \frac{1}{2}) = (\frac{1}{2}, \frac{3}{2})$.

$$B_\mathbb{C}(1 + i, 1) = \{ x \in \mathbb{C} \mid |x-(1+i)| < 1 \}$$

is the set of all points in $\mathbb{C}$ that are less than $1$ units away from $1 + i$. 

The diagram shows the open ball $B_\mathbb{C}(1 + i, 1)$ as a circle in the complex plane with radius $1$ centered at $1 + i$.
An open set in $F$ is a (possibly infinite) union of open balls.

\[ \text{e.g.} \quad \text{In } \mathbb{R}, \quad (a, b) = B\left(\frac{a+b}{2}, \frac{|a-b|}{2}\right) \quad \text{is always open} \]

- $\emptyset$ is open because its the empty union!
- $(a, b) \times (c, d)$ is open, but we have to use infinitely many balls!

**Challenge** Find an explicit collection of balls whose union is $(a, b) \times (c, d)$.

**Prop** Every nonempty open set contains infinitely many points.

**Pf** For each integer $n \geq 2$, $a + r/n \in B(a, r)$. □

**Cor** \( \{a\} \subseteq F \) is not open.

**Thm** 1. $\emptyset$ & $F$ are open  
2. Arbitrary unions of open sets are open  
3. Finite intersections of open sets are open 

i.e. intersections of finitely many

**Aside** This means that $F$ with its open subsets is a topological space.

**Pf Reading** □

**But note:** $\bigcap_{n \in \mathbb{Z}^+} B(a, 1/n) = \{a\}$, so finite intersections is essential.
**Definition:** A set $A \subseteq \mathbb{F}$ is **closed** if it is the complement of an open set $U \subseteq \mathbb{F}$: $A = \mathbb{F} \setminus U$.

*Example:* $\mathbb{F} \setminus B(a, r)$ is closed.

- $D(a, r) = \{ x \in \mathbb{F} \mid |x - a| \leq r \}$ is closed
  - disk of radius $r$ centered at $a$.

- $(b, \infty) = \bigcup_{n=1}^{\infty} B(a, 1)$
  - open

- $C \setminus \mathbb{C}$

  open by sim argument to
Defn. The closure of a set $A \subseteq F$, denoted $\overline{A}$, is the smallest closed subset of $F$ containing $A$.

e.g. $\overline{B(a, r)} = D(a, r)$

Defn. $A \subseteq F$ is dense if $\overline{A} = F$.

More generally, if $S \subseteq F$ is closed, say $A \subseteq S$ is dense in $S$ if $\overline{A} = S$.

Thm. $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$.

Defn. Let $A \subseteq F$. A point $a$ is in the interior of $A$ if $\exists r > 0$ such that $B(a, r) \subseteq A$. A point $a$ is on the boundary of $A$ if $\forall r > 0$, $B(a, r) \cap A \neq \emptyset$ and $B(a, r) \cap (F \setminus A) \neq \emptyset$.

$\text{Int}(A) = \{\text{interior pts of } A\}$

$\text{Bd}(A) = \{\text{boundary pts of } A\}$

Thm. $\forall A \subseteq F$, $\overline{A} = A \cup \text{Bd}(A)$

Pf. Reading.

So this says $\overline{\mathbb{Q}} = \mathbb{R} \Rightarrow \forall x \in \mathbb{R} \setminus \mathbb{Q}$, $B(x, r) \cap \mathbb{Q} \neq \emptyset$,

and $B(x, r) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$ for all $r > 0$.

In particular, $\forall r > 0$ and any $x \in \mathbb{R}$, there is always a rational number within $r$ of $x$. 