Then $F$ an ordered field, let

$F^+ = \{ x \in F \mid x > 0 \}$, $F^- = \{ x \in F \mid x < 0 \}$.

1. $x \in F^+ \iff -x \in F^-$ and $x \in F^- \iff -x \in F^+$

2. $1 \in F^+$.

If (1) $x \in F^+ \iff 0 < x$, add $(-x)$ to both sides:

$0 + (-x) < x + (-x)$

$-x < 0$.

Reversing these steps, $-x < 0 \Rightarrow x > 0$.

Other statement: similar.

(2) Note $1 \notin F$. Assume for contradiction $1 \in F^+$.

Then, by trichotomy, $1 \in F^-$.

Then $1 < 0 \Rightarrow -1 > 0$. Multiply by $-1$:

$(-1)(-1) > 0 \cdot (-1)$

$1 > 0$

This contradicts trichotomy $\Box$.
**Absolute value**  

**F-ordered field**

**Defn** \[ \mathbb{F} \rightarrow \mathbb{F} \]

\[ |x| = \begin{cases} 
  x & \text{if } x > 0 \\
  -x & \text{if } x < 0 \\
  0 & \text{if } x = 0 
\end{cases} \]

The "Standard" theorems about absolute value hold in an arbitrary ordered field.

In particular, the triangle inequality holds:

\[ |x + y| \leq |x| + |y| \]

**Defn** \[ S \subseteq \mathbb{F} \text{ then, if it exists, } \sup(S) \]

is the least upper bound of \( S \); \( \inf(S) \)

is the greatest lower bound of \( S \).

**e.g.** \[ S = \{ x \in \mathbb{Q} \mid 0 < x < 2 \} \]

\[ \sup(S) = 2 \]

\[ \inf(S) = 0 \]

\[ S = \{ \frac{1}{n} \mid n \in \mathbb{N} \cap \mathbb{F} \} \]

\[ \sup(S) = 1 \]

\[ \inf(S) = 0 \]

\( \mathbb{Q} \) is Archimedean
**Def.** An ordered field $F$ is **complete** if for every nonempty bounded above subset $S \subseteq F$, $\sup(S)$ exists as an elt of $F$.

**Prop.** If $F$ is complete and $S \subseteq F$ is bounded below, then $\inf(S)$ exists as an elt of $F$.

If $-S = \{-s \mid s \in S\}$ is bounded above, so it has a $\sup(-S) = -A$. Then $\inf(S) = -A$.

What is completeness? A Not having holes in the “number line” for $F$:

![Number line diagram](diagram)

Completeness says $\sup(-S)$ is one of the blue dots (elts of $F$).
Then $\mathbb{Q}$ is not complete.

Note $\mathbb{R}$, the real numbers, is the smallest complete ordered field containing $\mathbb{Q}$.

It suffices to find $S \subseteq \mathbb{Q}$ bdd above w/out a supremum in $\mathbb{Q}$.

Take $S = \{ x \in \mathbb{Q} \mid 0 < x, \ x^2 < 2 \}$.

Claim 1: $S \neq \emptyset$ and bdd above.

Claim 2: If $U = \sup(S)$, then $U^2 = 2$

Claim 3: $\forall q \in \mathbb{Q}, \ q^2 \neq 2$.

Claim 1: $1 \in S \neq \emptyset$.

We claim 3 is an upper bd of $S$.

To see this, note that $x, y > 0$ & $x^2 < y^2$, then $x < y$.

| By contrapositive: assume $x \geq y$ & $xy > 0$. | Mult by $x$: $x^2 \geq xy$ |
| Mult by $y$: $xy \geq y^2$ |
| Transitivity: $x^2 \geq y^2$. |
Now if \( x \in S \) so \( 0 < x < x^2 < 2 \).

Then \( 2 < 3^2 = 9 \) so we have \( x^2 < 3^2 \Rightarrow x < 3 \)
i.e. \( 3 \) is an upper bound of \( S \).

Claim 3 Proof by contradiction: assume for

contradiction \( \exists q \in \mathbb{Q} \) s.t. \( q^2 = 2 \).

Write \( q = \frac{a}{b} \) in least terms (\( a \& b \) share no
common factors). Know \( \frac{a^2}{b^2} = 2 \) so \( a^2 = 2b^2 \),
i.e. \( a^2 \) is even. Thus \( a \) is even \((b/c \ \text{odd}^2 = \text{odd})\). So we can write \( a = 2l \) for

some integer \( b \).

\[
(2l)^2 = 2b^2
\]
\[
4l^2 = 2b^2
\]
\[
2l^2 = b^2
\]

i.e. \( b^2 \) is even so \( b \) is even.

This contradicts \( \frac{a}{b} \) in least terms (\( a, b \)
share a factor of \( 2 \)) \( \blacksquare \)
Thm. There exists a unique complete ordered field $\mathbb{R}$.

Any two complete ordered fields $F$, $F'$ are in fact "ordered isomorphic."

I.e. $\exists f : F \rightarrow F'$

r.t.

- $f(x+y) = f(x) + f(y)$
- $f(x \cdot y) = f(x) \cdot f(y)$
- if $x < y$, then $f(x) < f(y)$

and $f$ is bijective.

Back to Claim 2: If $S = \{ x \in \mathbb{Q} \mid 0 < x, x^2 < 2 \}$ and $U = \text{sup}(S)$, then $U^2 = 2$. Proving this will complete our proof that $\mathbb{Q}$ is not complete!

Lemma

1. If $0 < x \in \mathbb{Q}$ & $x^2 < 2$, then $\exists y \in \mathbb{Q}$ r.t. $x < y$ & $y^2 < 2$.
2. If $0 < x \in \mathbb{Q}$ & $x^2 > 2$, then $\exists y \in \mathbb{Q}$ r.t. $x > y$ & $y^2 > 2$. 
Note that the lemma will imply: If \( U < 2 \), then

\[ \exists y \in \mathbb{Q}, \; y < 2, \; y > U \; \text{contradicting} \; U \; \text{an upper bd of} \; S. \]

Thus \( U^2 > 2 \). Similarly, \( \circ \) implies \( U^2 < 2 \), so by trichotomy, \( U^2 = 2 \).

It remains to prove the lemma:

If Lemma \( \circ \), assume \( 0 < x \in \mathbb{Q}, \; x^2 < 2 \). Then

\[ 2x+1 > 0 \; \& \; 2-x^2 > 0 \; \text{so} \; \frac{2x+1}{2-x^2} > 0. \]

Let \( N \) be an integer st. \( N > \frac{2x+1}{2-x^2} \) (use Archimedean property of \( \mathbb{Q} \)). Then

\[ \frac{N}{2x+1} > \frac{1}{2-x^2} \]

\[ \Rightarrow \frac{2x+1}{N} < 2-x^2 \]

\[ \Rightarrow \frac{2}{N} x + \frac{1}{N} < 2 - x^2 \]

\[ \Rightarrow x^2 + \frac{2}{N} x + \frac{1}{N} < 2 \]

Since \( \frac{1}{N^2} = \left( \frac{1}{N} \right)^2 < \frac{1}{N} \), get \( (x + \frac{1}{N})^2 = x^2 + \frac{2}{N} x + \frac{1}{N^2} < x^2 + \frac{2}{N} x + \frac{1}{N} < 2 \), i.e. \( (x + \frac{1}{N})^2 < 2 \).

Thus \( y = x + \frac{1}{N} \in \mathbb{Q}, \; x < y, \; y^2 < 2 \), as desired.

The proof of \( \circ \) is similar. \( \Box \) Not complete!