Order on $\mathbb{N}$

$0 = \emptyset \quad 1 = 0^+ = 0 \cup \{0\} \quad 2 = 1^+ = 1 \cup \{1\}$

$= \emptyset \cup \{\emptyset\} \quad = \{\emptyset\} \cup \{\emptyset\} \quad = \{\emptyset, \{\emptyset\}\}$

$= \emptyset \quad = \emptyset \quad = \{\emptyset, \{\emptyset\}\}$

$0 \in 1$

$3 = 2^+ = 2 \cup \{2\}$

$= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$

$= \{\emptyset, \{\emptyset\}, \emptyset, \{\emptyset, \{\emptyset\}\}\}$

$0, 1, 2 \in 3 \quad \ldots \quad 0, 1, 2, \ldots, n-1 \in n$

Defn: For $m, n \in \mathbb{N}$, $m < n$ iff $m \in n$

Notes: $<$ is a relation on $\mathbb{N}$ ("$\geq$"")

- not reflexive
- not symmetric
- transitive

If $m < n$, and $n < p$, then $m < p$.

If $T = \{p \mid m \in n \land n \in p \Rightarrow m \in p\}$

Show $T$ inductive $\Rightarrow T = \mathbb{N}$. □
**Trichotomy.** If $m, n \in \mathbb{N}$, then exactly one of the following holds:

1. $m < n$
2. $m = n$
3. $n < m$.

**Defn.** $m \leq n$ iff $m < n$ or $m = n$

- $m > n$ iff $n < m$
- $m \geq n$ iff $n \leq m$

**Well-ordering.** Every non-empty subset of $\mathbb{N}$ has a least element.

$s \in S \subset \mathbb{N}$ is the least elt of $S$ when $s \leq n$ for every $n \in S$.

If $T = \ldots$ notes!
The integers \( \mathbb{N} \) lacks a notion of "debt." Can often (half of the time?) subtract natural numbers:

\[
7 - 4 = 3
\]
saying that 3 is the number such that

\[
3 + 4 = 7,
\]

i.e. sometimes we can solve

\[
7 = 3 + m \quad \text{w/} \quad m \in \mathbb{N}
\]

\[
2 = 5 + m \quad \text{no solution in} \quad \mathbb{N}!
\]

Idea: Record \((2,5)\) \(\in \mathbb{N} \times \mathbb{N}\) and use it as a placeholder for some object \(x\) satisfying

\[
2 = 5 + x
\]

In other words, let's write \((2,5)\) for \(2 - 5\)

Question: Doesn't the same \(x\) satisfy

\[
1 = 4 + x
\]
In order to eliminate redundancy, we define an equivalence relation \( \sim \) such that
\[(2, 5) \sim (1, 4) .\]

Define a relation \( \sim \) on \( \mathbb{N} \times \mathbb{N} \) so that
\[(a, b) \sim (a', b') \quad \text{iff} \quad a + b' = a' + b. \]

Notation \([a, b]\) is the equivalence class of \((a, b)\).

i.e. \([a, b]\) = \{ \((a', b') \in \mathbb{N} \times \mathbb{N} \mid (a, b) \sim (a', b')\} .\]

Define the integers \(\mathbb{Z}\) as \([\{(a, b) \mid (a, b) \in \mathbb{N} \times \mathbb{N}\}] .\)