Goals

- What is a number?
- Prove that $1 + 1 = 2$.
- An abstract "value" or "quantity" representing "how many."
- Provide quantitative measurements.
- The measurements have "units" — things in the same category — but numbers don't capture quantity w/o reference to category.

Inductive sets

$\emptyset$ $\{\emptyset\}$ $\{\emptyset, \{\emptyset\}\}$ $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

0 1 2 3

0 $\{\emptyset\}$ 1 $\{\emptyset, \{\emptyset\}\}$ 2 $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
Defn. For any set $S$, the successor of $S$ is

$$S^+ = S \cup \{S\}.$$  

Note. Self-referentially builds in "1 more" element.

$0 = \emptyset$, $1 = 0^+$, $2 = 1^+$, $3 = 2^+$, $\ldots$

Defn. A set $J$ is inductive if:

1. $\emptyset \in J$
2. For all $n \in J$, $n^+ \in J$.

Axiom. There exists an inductive set.

Thm. Suppose $S$ is a set containing inductive subsets. Then the intersection of these subsets is also inductive.

Pf. Any in. subset contains $\emptyset$ by (1), thus their intersection contains $\emptyset$ as well. If $n$ is in the intersection, then $n$ belongs to all the in. subsets. By (2), $n^+ \in$ all in. subsets $\Rightarrow n^+ \in$ intersection of in. subsets.$\square$
Defn Let $J$ be some inductive set. Let $N$ be the intersection of all inductive subsets of $J$. This is the set of natural numbers.

Remk $N$ is the smallest (under $\subseteq$) inductive subset of $J$.

Remk The defn depend on $J$ — $N$ nonetheless is independent of $J$.

Thm $0$ is not the successor of any elt of $N$.

Every $n \notin \inf$ is the successor of some elt of $N$.

If $0 = \emptyset$ is clearly not the successor $S^+ = S \cup \{S\}$ of any $S$. Now assume for contradiction that $n \in N \setminus \inf$ is not the successor of any elt of $N$.

Then $N \setminus \inf$ is an inductive subset of $N$.

This is a contradiction by minimality of $N$ amongst inductive subsets. $\square$
**Induction Theorem** Suppose $P$ is a property depending on (some) elements of $\mathbb{N}$. Suppose

1. $P(0)$ is true
2. $\forall n \in \mathbb{N}, \left[ P(n) \implies P(n^+) \right]$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

**Proof** Let $T = \{ n \in \mathbb{N} \mid P(n) \text{ is true} \}$. Suffices to show $T$ is an inductive subset of $\mathbb{N}$, whence $T = \mathbb{N}$ & we're done. $\emptyset \in T$ since $P(0)$ is true. $\emptyset \subset \mathbb{N}$ & we're done. Now suppose $n \in T$ so that $P(n) \implies P(n) = T$. Then $P(n^+) = T$ by (2). Thus $n^+ \in T$, proving $T$ is inductive. \[ \square \]

Thus $\forall n \in \mathbb{N}$, $0 \in n^+$. \[ \square \]