Convex Polytopes
of
Permutation Matrices

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Abstract

Given a permutation group $G$ in $S_n$, we can construct the set of permutation matrices of $G$ as a set of $n \times n$ matrices with exactly one 1 per row and column, where each matrix is defined using an element of $G$. We can take the convex hull of these matrices, thought of as points in $\mathbb{R}^{n^2}$ space, to form the $G$-permutation polytope. We find the projection of this polytope from $\mathbb{R}^{n^2} \rightarrow \mathbb{R}^n$ defined by each permutation matrix $X$ acting on a vector $a = (a_1, ..., a_n)$. We call this the $G$-orbit polytope. We find properties of these two polytopes for the symmetric, alternating, and dihedral groups.
Chapter 1

Introduction

1.1 Permutation groups

A permutation is a bijection which takes a set $A$ to itself. A permutation group of a set $A$ is a set of permutations of $A$ that forms a group under function composition. We will be looking at groups of permutations of a nonempty, finite set $A$ of the form $\{a_1, a_2, ..., a_n\}$. Permutations of finite sets are given by an explicit listing of each element of the domain and its corresponding functional value. For example, we define a permutation $\alpha$ of the set $\{1,2,3,4\}$ by specifying

$$\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 1, \alpha(4) = 4.$$ 

A more convenient way to express this correspondence is to write $\alpha$ in cyclic form. Cyclic form is always written as a product of $m$-cycles: elements $(a_1, a_2, ..., a_m)$ where $a_1$ is permuted to $a_2$ and so on until $a_m$ is permuted to $a_1$. In cyclic notation, $\alpha = (1\ 2\ 3)$, a 3-cycle. To take products of $m$-cycles, move from right to left from one cycle to the next, where any missing symbol is left where it is. For example, take $\beta = (3\ 2\ 1)(5\ 4)$. Start with 1 in the right cycle; however, 1 does not appear in this cycle, so $(5\ 4)$ fixes 1. Move on to the second cycle. It tells you to move 1 to 3; $(3\ 2\ 1)$ sends 1 to 3. Continuing in this way, the numbers $\{1,2,3,4,5\}$ are permuted
to \((2, 3, 1, 5, 4)\), in order. We could just as easily have used some list of 5 elements \(\{a, b, c, d, e\}\). Under the same action \(\beta\), this would be permuted to \((b, c, a, e, d)\). Two cycles are disjoint if they share no elements in common. For example, \((123)\) and \((45)\) are disjoint, \((123)\) and \((25)\) are not disjoint. Every permutation can be written as a product of disjoint cycles.

Define \(S_n\) to be the symmetric group of order \(n\). The symmetric group of order \(n\) is the set of all permutations of the \(n\)-element set \(A\). A standard counting argument shows that \(S_n\) has \(n! = n(n - 1)(n - 2) \cdots 2 \cdot 1\) elements. Here are the elements of \(S_4\):

\[
\begin{array}{ccccccccc}
(1) & (1234) & (1324) & (14)(23) & (12)(34) & (13)(24) \\
(12) & (34) & (13) & (24) & (14) & (23) \\
(123) & (234) & (132) & (142) & (1243) & (143) \\
(1324) & (1432) & (243) & (134) & (124) & (1423) \\
\end{array}
\]

One subset of \(S_n\) consists of all of the even permutations of \(n\) objects, which we now describe. Remember that we represented a permutation of \(n\) objects as a product of \(m\)-cycles. We can rewrite each \(m\)-cycle as a product of 2-cycles. For example, \((1234) = (12)(13)(14)\). This decomposition is not unique, and we can even decompose to different numbers of 2-cycles. However, we always decompose to either an even or an odd number of decompositions. If a permutation can be decomposed to an even number of 2-cycles, then it is an even permutation. The set of even permutations forms a group. This subgroup of \(S_n\) is called \(A_n\), the alternating group of degree \(n\). For an example, look at the elements of \(A_4\), the set of even permutations of 4 elements. Notice that exactly half of the elements of \(S_n\) are in \(A_n\).

\[
\begin{array}{ccccccccc}
(1) & (12)(34) & (13)(24) & (14)(23) \\
(123) & (134) & (243) & (142) \\
(132) & (234) & (124) & (143) \\
\end{array}
\]

Another subset of \(S_n\) consists of all of the symmetries of a regular \(n\)-gon. This subset consists of the \(2n\) elements of \(S_n\) which rotate or reflect some \(n\)-gon while preserving
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its position in space. In general, we can say $D_n = \langle \rho, \phi \mid \rho^n = \phi^2 = e, \rho\phi = \phi\rho^{n-1}\rangle$, where $\rho$ is a rotation of an $n$-gon by $360/n$ degrees, $\phi$ is a reflection about a line of mirror symmetry, and $e$ is the identity permutation, where no points are permuted. In other words, $D_n$ is the set of all products of various powers of $\rho$ and $\phi$ but we can use the relation $\phi^n = \rho^2 = (1)$. This subset actually forms a subgroup, called $D_n$, the dihedral group of order $2n$. For example, take a 4-gon, commonly known as a square. We can rotate the square in increments of 90 degrees without changing the square’s position in space. We can also flip the square like a pancake-horizontally, vertically, and diagonally. When we label the four corners of the square in a clockwise manner with $1, 2, 3, 4$. $\rho$ and $\phi$ can be represented in cyclic notation as $(1234)$ and $(12)(34)$ respectively. Repeatedly combining these two actions with themselves or with each other give all of the possible elements of $D_4$. Here are the elements of $D_4$, the symmetries of the square. Note there are 8 elements:

$$(1), (12)(34), (13)(24), (24), (1234), (13), (14)(23), (1432).$$

When $n$ is odd, we have an odd dihedral group. A permutation in an odd dihedral group fixes either 0 points or 1 point. This is obvious, because the rotations change all points, and the line of mirror symmetry of a regular odd sided $n$-gon goes through exactly one vertex. Therefore a reflection through this line fixes one point. When $n$ is even, we have an even dihedral group. A permutation in an even dihedral group fixes either 0 points or 2 points. Clearly, the line of mirror symmetry of an regular even sided $n$-gon will contain either zero or two vertices. Thus, a reflection through this line will fix either zero or two points.
1.2 Polytopes

A point set is convex if for any two points $x$ and $y$ in the point set, the straight line segment

$$[x, y] = \{ \lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1 \}$$

between them is also in the point set. Every intersection of convex sets is convex. The convex hull of a set of points is the “smallest” convex set containing the points. Specifically, for any point set $K$, the convex hull of $K$ is constructed by taking the intersection of all convex sets that contain $K$:

$$\text{conv}(K) := \bigcap \{ K' \subset \mathbb{R}^d \mid K \subset K', K' \text{ is convex} \}.$$  

If $K$ is a finite set, this convex hull will be called a V-polytope.

Another creation is the $H$-polyhedron, which uses the concept of halfplanes. A halfplane is just as it sounds: all of the area to one side of a defining cut; that is, those points $x \in \mathbb{R}^n$ defined by $c \cdot x \leq c_0$ for some constant $c_0$ and some $c \in \mathbb{R}^n$. An $H$-polyhedron $P$ is formed by taking the intersection of finitely many closed
halfplanes in some $\mathbb{R}^d$:

$$P = P(A, z) = \{x \in \mathbb{R}^d \mid Ax \leq z\} \quad \text{for some } A \in \mathbb{R}^{m \times d}, \ z \in \mathbb{R}^m.$$ 

An $H$-polyhedron that is bounded in the sense that it does not contain a ray $\{x + ty \mid t \geq 0\}$ for any $y \neq 0$ is called an $H$-polytope. It turns out that every $V$-polytope is an $H$-polytope, and vice-versa (for the proof, see [Ziegler], p. 29). From now on, we will use the word polytope to mean $V$-polytope or $H$-polytope.

### 1.2.1 Faces

We will be looking at properties of faces of polytopes, defined to be the intersections of the polytope $P$ with hyperplanes for which the polytope is entirely contained in one of the two halfspaces determined by the hyperplane. In other words, $F$ is a face of $P$ if

$$F = P \cap \{x \in \mathbb{R}^d \mid c \cdot x = c_0\}$$

where $c \cdot x \leq c_0$ is satisfied for all points $x \in P$. 

Figure 1.2: A $V$-polytope and an $H$-polytope
To define the dimension of a face $F$, we first introduce the notion of the affine hull of $F$: Pick a point $p \in F$, and let $L$ be the linear space spanned by $F - p := \{q - p \mid q \in F\}$. Then the affine hull of $F$, denoted $\text{aff}(F)$, is $p + L$, i.e., the smallest affine space containing $F$. The dimension of $\text{aff}(F)$ is defined to be the dimension of $L$. Say $v_1, ..., v_k$ is a basis for $L$. Thus, every point $q \in F$ can be written as $q = p + \sum a_i v_i = (1 + \sum a_i)p + \sum a_i(v_i - p_i)$. Thus we have found points $x_1 := p, x_2 := v_1 - p, ..., x_{k+1} := v_k - p$ of $F$ such that $\text{aff}(F) = \{\sum \lambda_i x_i \mid \sum \lambda_i = 1\}$.

**Definition 1.1** The dimension of a face is the dimension of its affine hull, $\dim(F) := \dim(\text{aff}(F))$.

In a $d$-dimensional polytope $P$, the faces of $P$ of dimension 0 are the vertices of $P$. The edges of $P$ are those faces of dimension 1. Facets are the $d - 1$ dimensional faces. In general, $P$ has a set of faces of every dimension $k, 0 \leq k \leq d$. A face of codimension $k$ has dimension $d - k$.

Consider the square in $\mathbb{R}^2$ created by the halfplanes $x \geq 0, y \leq 1, x \leq 1$, and $y \geq 0$. Then the vertices are the points $(0,0), (1,0), (1,1)$, and $(0,1)$. The edges are the intersections of the square with the lines $x = 0, x = 1, y = 0$, and $y = 1$. In this case, the edges are the facets of this polytope. The 2-dimensional face is the entire square. Two polytopes are considered combinatorially equivalent if there is a bijection between their faces that preserves the inclusion relation among faces. To aid the combinatorial analysis we can construct the face lattice of a polytope. Let $S$ denote the set of faces of a polytope $P$. The inclusion relation among faces defines a partial ordering on $S$. Under this relation, $S$ has a unique maximal element, namely $P$ itself, and a unique minimal element, $\emptyset$, the empty set. Further, every two faces are minimally contained in a unique face and contain a unique maximal subface. Thus, $S$ forms what is called a lattice. It turns out that if $F$ is a $k$-face, then the length of any maximal totally ordered subset of $S$ having maximal element $F$ has
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length $k+1$. For more on this topic, see [Ziegler]. Rephrasing what was said earlier, two polytopes are combinatorially equivalent if their face lattices are isomorphic.

1.2.2 Simplicial polytopes

The convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$, where $n \geq d$, is called a $d$-simplex; thus, the $d$-simplex is a polytope of dimension $d$ with $d+1$ vertices. In two dimensions, a triangle is a simplex. A tetrahedron is a three dimensional simplex. A polytope $P$ is simplicial if every facet is a simplex. For example, the icosahedron is a three dimensional simplicial polytope, since each of its facets are triangles, which are simplices. Every facet of a simplicial polytope has $d$ vertices, and every $k$ face has $k+1$ vertices for $k \leq d-1$ (for a proof, see [Ziegler].)
1.3 Polytopes arising from permutation groups

1.3.1 Permutation matrices

Given a permutation group $G$ inside $S_n$, we define the set of permutation matrices of $G$ by

$$(X_\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise} \end{cases}$$

for all $\sigma$ in $G$. The $n \times n$ matrices $X_\sigma$ are 0/1-matrices with exactly one 1 per row and column.

For an example of a permutation matrix, consider $\sigma = (123)$ in $S_3$. Then $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. The permutation matrix associated with $\sigma$ is

$$X_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

For another example, look at $\sigma = (123)$ in $S_4$. Again, $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, but now we also have $\sigma(4) = 4$, which means 4 was not affected by the permutation. Whenever a number does not appear in a permutation, it is not affected by the permutation and a 1 appears on the diagonal:

$$X_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The set of permutation matrices of the alternating group $A_n$ consists of exactly half of all the permutation matrices: those matrices with determinant equal to $+1$. This is because we only take those matrices which can be obtained from the identity matrix with a even number of row transpositions. From linear algebra, we know that each row transposition changes the sign of the determinant. Therefore, a permutation matrix is even only if it has determinant equal to $+1$. 


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1.3.2 How to get a polytope in $\mathbb{R}^{n^2}$ from a permutation group

Each of the permutation matrices of the set $G$ can be flattened, its rows listed one after another, and considered to be a point in $\mathbb{R}^{n^2}$. The convex hull of this set forms the $G$-polytope, or permutation polytope,

$$P(G) := \text{conv} \{X_\sigma \mid \sigma \in G\}.$$ 

1.3.3 The structure of $P(G)$

In this section we will look at the vertices, edges, and facets of the permutation polytope $P(G)$.

**Theorem 1.2** Each $X_\sigma$ is a vertex of $P(G)$

**Proof** Consider maximizing the inner product $\langle X, X_\sigma \rangle$ as $X$ varies over $P(G)$.

$$\langle X, X_\sigma \rangle = \sum_{1 \leq i, j \leq n} x(i, j) x_\sigma(i, j)$$

$$= \sum_{1 \leq j \leq n} x(i, \sigma(i)) \leq n$$

with equality if and only if $X(i, \sigma(i)) = 1$ for all $i$. That is, $\langle X, X_\sigma \rangle$ is maximal exactly when $X$ equals $X_\sigma$. So $X_\sigma$ is a vertex of the polytope. $\square$

To describe the edges of $P(G)$, we can use the following theorem, known as the cycle-decomposition theorem. It tells us when the line between two vertices $X_\sigma$ and $X_\pi$ is an edge.

**Theorem 1.3** Cycle Decomposition The line segment $\{X_\sigma, X_\pi\}$ between the vertices $X_\sigma$ and $X_\pi$ is an edge of the polytope constructed from the convex hull of matrices $X_\sigma$ such that $\sigma$ is in a group $G$, if and only if the cycle decomposition of $\sigma^{-1} \pi$ cannot be factored into two non-trivial parts, both of which are elements of $G$. 

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PROOF: It suffices to show the theorem with respect to the vertices $X_\pi$ and $X_e$, where $e$ is the identity permutation. We need to show that the line segment between $X_\pi$ and $X_e$ is an edge of the polytope if and only if the cycle decomposition for $\pi$ cannot be factored into a product of two elements of the group. If the cycle decomposition factors as $\pi = \pi_1 \pi_2$ then $1/2X_e + 1/2X_\pi = 1/2X_{\pi_1} + 1/2X_{\pi_2}$. For example, take $\pi = (321)(45)$. Then $\pi_1 = (321)$ and $\pi_2 = (45)$ and we see that:

$$\frac{1}{2}X_e + \frac{1}{2}X_{(321)(45)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}X_{(321)} + \frac{1}{2}X_{(45)}.$$ But we know from geometry that two vertices (extreme points) $u$ and $v$ of a convex polytope determine an edge if and only if no point $cu + (1-c)v$ with $0 \leq c \leq 1$ on the line segment joining $u$ and $v$ can be represented as a nontrivial convex combination of two points of the polytope at least one of which does not belong to the line segment. Hence, if $\pi$ can be factored into two nontrivial elements of the group, a point of the line segment between $e$ and $\pi$ can be written as a convex combination
of two other points in the group, and so is not an edge.

If the line segment between $X_e$ and $X_\pi$ is not an edge, we will now show that the cycle decomposition for $\pi$ factors nontrivially as $\pi = \sigma \tau$ where both $\sigma$ and $\tau$ are in $G$.

Let $X = \frac{1}{2}X_e + \frac{1}{2}X_\pi$. If the line segment between $X_e$ and $X_\pi$ is not an edge, then we can write $X$ as a positive convex combination

$$X = \sum_{\sigma \in G} \lambda_\sigma X_\sigma, \quad \lambda_\sigma \geq 0, \quad \sum_\sigma \lambda_\sigma = 1,$$

where some $\lambda_\sigma$ is nonzero for $\sigma \notin \{e, \pi\}$. Fix some such $\sigma$. Since we are taking nonnegative combinations of matrices with nonnegative entries, whenever a zero appears in an entry for the matrix $X$, then a zero must appear in the corresponding entry in $X_\sigma$. Since there are at most two nonzero entries on each row of $X$, this means that if $\sigma(i) \neq i$, then $\sigma(i) = \pi(i)$.

We now show that every factor in the cycle decomposition for $\sigma$ is a cycle in the cycle decomposition for $\pi$. Thus the cycle decomposition for $\pi$ can be factored into two parts: $\sigma$ and the product of the remaining cycles, which we denote by
τ. Since π and σ are in G and τ = σ⁻¹π, it follows that τ is in G, and we will be done. To accomplish this, take \( i_1 \) such that \( σ(i_1) \neq i_1 \). The remarks in the previous paragraph show that in this case \( σ(i_1) = π(i_1) \). Let \( (i_1, \ldots, i_k) \) be the corresponding cycle in the cycle decomposition of π. We need to show that this cycle occurs in the decomposition for σ, as well. Suppose \( σ(i_m) = π(i_m) = i_{m+1} \) for some \( m < k \). By remarks in the previous paragraph, if \( σ(i_{m+1}) \neq π(i_{m+1}) \), then \( σ(i_{m+1}) = i_{m+1} \). However, then we have \( σ(i_m) = σ(i_{m+1}) \), contradicting the fact that σ is a permutation. This completes the proof. □

1.3.4 Orbits

Given a permutation group \( G \) in \( S_n \) and a point \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), define the orbit of \( a \) under \( G \) to be the set of all points

\[
x_σ := \begin{pmatrix} a_{σ(1)} \\ \vdots \\ a_{σ(n)} \end{pmatrix} = X_σ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}
\]

for \( σ \in G \). The convex hull of the orbit defines the orbit polytope

\[
O(G, a) = \text{conv} \{ x_σ \mid σ \in G_n \}
\]

Alternatively, \( O(G, a) \) is the image of \( P(G) \) under the projection \( \mathbb{R}^{n^2} \to \mathbb{R}^n \) defined by

\[
X \mapsto X \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}
\]

thinking of \( X \in \mathbb{R}^{n^2} \) as an \( n \times n \) matrix.
1.4. SUMMARY

In the following chapters, we will find that the structure of $O(G, a)$ depends on the vector $a$ for certain groups. In general, we only know about the vertices of the projected permutation polytope.

**Theorem 1.4** Each point $x_\sigma$ is a vertex of $O(G, a)$.

**Proof** Since $O(G, a)$ is the convex hull of the set $x_\sigma$, we know at least one of these points is a vertex. Pick a vertex $x_\sigma$. Let $x_\pi$ be any other point in the orbit. Then $\sigma \pi^{-1}$ defines a linear isomorphism from $\mathbb{R}^n \rightarrow \mathbb{R}^n$ sending $O(G, a)$ to itself and sending $x_\pi$ to $x_\sigma$. Hence $x_\pi$ is a vertex, too. \qed

1.4 Summary

1.4.1 The Symmetric Group

The convex hull of the group of all $n \times n$ permutation matrices is called the Birkhoff polytope, a polytope of dimension $(n - 1)^2$ with each matrix as a vertex, giving $n!$ vertices. We can describe this polytope with inequalities representing the hyperplanes which define it, (Theorem 2.1). The cycle decomposition theorem tells us how to find the edges of this polytope, (Theorem 2.3).

We proceed to take the projection of the Birkhoff polytope to get the permutahedron, which has the permutations of the vector $a = (a_1, \ldots, a_n)$ under permutations in $S_n$ as its vertices. We find its dimension, (Theorem 2.4). We can describe this polytope using inequalities which define the hyperplanes framing it, (Theorem 2.7). We realize that the face lattice of the permutahedron is isomorphic to lattice of chains of subsets of $[n] := \{1, 2, \ldots, n\}$, (Proposition 2.10). With this information, we can determine the $f$-vector, which tells us how many faces there are of each dimension, and we know how to find such faces, (Theorem 2.12). Finally, we determine which vertices are adjacent to one another, finding that the vertices adjacent
to a given vertex are those vertices whose coordinates differ from the given vertex by a single transposition, (Theorem 2.14).

1.4.2 The Alternating Group

In this chapter we look at the alternating polytope, an \((n-1)^2\) dimensional polytope with \(n!/2\) vertices, (Theorem 3.1). The cycle decomposition theorem tell us that the line segment \(\{X_\sigma, X_\pi\}\) in the alternating polytope is an edge if and only if the cycle decomposition of \(\sigma^{-1}\pi\) consists of exactly 1 cycle of odd length, or exactly two cycles of even length, (Theorem 3.2). The projection of the alternating polytope yields the alternahedron, which can also be constructed by cutting vertices off of the permutahedron. We give the inequality description, and its dimension, (Theorems 3.3 and 3.4). We give data for several alternahedra and ask questions which could lead to further research.

1.4.3 The Dihedral Group

In this chapter we look at the dihedral polytope, and find that its dimension changes depending on the parity of \(n\), (Theorem 4.1). The cycle decomposition theorem tells us that every vertex is connected to every other vertex with an edge of the polytope when \(n > 4\), (Theorem 4.2). We also find that the dihedral polytopes are simplicial for odd \(n\), (Theorem 4.3). Our data suggest several conjectures which remain to be proved.

We call the projection of the dihedral polytope the dihedron, and find its dimension, (Theorem 4.7). We also find that the dihedron is not unique for generic \(a\). Depending on the vector we choose to permute, we can get drastically different polytopes. We look at some possibilities and ask more questions.
1.4.4 Questions

We present a list of questions which have come up in the duration of the thesis.
Chapter 2

The Symmetric Group

This chapter is mainly an exposition of theory from two sources: [Billera] and [YKK]. We have combined the ideas from both, as well as adding a few ideas of our own, to get a more complete theory than either of the others achieved alone.

The convex hull of the permutation matrices of the symmetric group, thought of as points in $\mathbb{R}^{n^2}$, notated $B_n := \text{conv} \{X_\sigma \mid \sigma \in S_n\}$, forms the Birkhoff polytope. We will now find several properties of this object.

2.1 The Birkhoff Polytope

**Theorem 2.1** $B_n$ is an $(n-1)^2$ dimensional polytope with $n!$ vertices having the following inequality description:

$$B_n = \{X = (x_{ij}) \in \mathbb{R}^{n^2} \mid x_{ij} \geq 0; 1 \leq i, j \leq n, \sum_{j=1}^{n} x_{ij} = 1$$

for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1, \ldots, n\}$$

Thus, $B_n$ consists of what are called doubly stochastic matrices: matrices with nonnegative entries and whose row and column sums are 1. We will call the right
hand side of the equality $C_n$. Before jumping into the proof for this theorem, observe
the following results about $C_n$:

**Lemma 2.2** The equations of $C_n$ satisfy $2n - 1$ independent linear equations.

**Proof:** In the case of $n = 3$ it is easy to represent the equations of $C_n$ with the
following matrix:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

Subtracting the bottom three rows from the first gives

$$
\begin{pmatrix}
0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

Adding rows two and three to row one gives

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

Subtract columns one through three from four through six, and seven through nine, in turn, to get

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

2.1. THE BIRKHOFF POLYTOPE

It is clear that this matrix has five linearly independent rows. This can be generalized to the matrix with 2n equations

$$\begin{pmatrix}
\vec{1} & \vec{0} & \vec{0} & \cdots & \vec{0} & 1 \\
\vec{0} & \vec{1} & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vec{0} & \cdots & \vec{0} & \vec{1} & \cdots & \vec{1} \\
I_n & I_n & \cdots & I_n & I_n & 1
\end{pmatrix}$$

where $I_n$ is the $n \times n$ identity matrix, $\vec{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Using row and column operations as before, this matrix reduces to

$$\begin{pmatrix}
\vec{0} & \vec{0} & \cdots & \cdots & \vec{0} & 0 \\
\vdots & \vec{1} & \vec{0} & \cdots & \cdots & \vdots \\
\vdots & \vec{0} & \cdots & \cdots & \vdots & \vdots \\
\vec{0} & \cdots & \vec{0} & \vec{1} & \cdots & \vec{1} \\
I_n & \vec{0} & \cdots & \vec{0} & \vec{0} & \vec{1}
\end{pmatrix}$$

leaving us with $2n - 1$ independent linear equations. □

Now we go on to prove theorem 1.1.

**Proof [Billera]:** Remember that each $X_\sigma$ is a vertex of $B_n$ by Theorem 1.2. We know that elements of $C_n$ satisfy $2n - 1$ independent linear equations by the previous lemma; therefore, the dimension of $C_n$ is $n^2 - (2n - 1) = (n - 1)^2$. Since each $X_\sigma$ is in $C_n$, it follows that $B_n \subset C_n$. To show $B_n = C_n$, use induction on $n$ to show each vertex of $C_n$ is a permutation matrix. If a matrix $X$ is a vertex of $C_n$ then a standard result from the theory of polytopes says that $X$ sits on at least $(n - 1)^2$ facets. Since the facet defining equations of $C_n$ have the form $x_{ij} = 0$, it follows that $X$ has at least $(n - 1)^2$ entries equal to zero. This implies that $X$ must have a row with $n - 1$ zeroes. So, $x_{ij} = 1$ for some $i$ and $j$. Without loss of generality, let $i = j = 1$. Deleting the first row and first column from $X$ leaves us with a
(n − 1) × (n − 1) matrix which we will call \( \tilde{X} \). We would like to show that \( \tilde{X} \) is a
vertex of \( C_{n-1} \). Then, by induction \( \tilde{X} \) is a permutation matrix, hence \( X \) was one,
as desired. Since \( X \) is a vertex, there exists some \( D = (d_{11}, \ldots, d_{mn}) \in \mathbb{R}^{n^2} \) such that
\( \max\{\langle Y, D \rangle \mid Y \in C_n\} = X \). This implies that \( \sum D_{ij}Y_{ij} \leq \sum D_{ij}X_{ij} \) for all \( Y \in C_n \).
Define \( \tilde{D} \in \mathbb{R}^{(n-1)^2} \) by removing the first “row” and “column” of \( D \), i.e., removing
\( d_{ii} \) and \( d_{i1} \) from \( D \) for \( i = 1, 2, \ldots, n \). If \( \tilde{X} \) were not a vertex of \( C_{n-1} \) then there
would exist \( \tilde{Y} \) in \( C_{n-1}, \tilde{Y} \neq \tilde{X} \), such that \( \langle \tilde{Y}, \tilde{D} \rangle \geq \langle \tilde{X}, \tilde{D} \rangle \). Define \( Y \in \mathbb{R}^{n^2} \) by
\[
Y := \begin{pmatrix}
1 & 0 \\
0 & \tilde{Y}
\end{pmatrix}
\]
It follows that \( Y \in C_n, \tilde{X} \neq Y \), and
\[
\langle Y, D \rangle = d_{11} + \langle \tilde{Y}, \tilde{D} \rangle \geq d_{11} + \langle \tilde{X}, \tilde{D} \rangle = \langle X, D \rangle.
\]
which contradicts the fact that \( X \) is a vertex. Thus, \( \tilde{X} \) is a vertex of \( C_{n-1} \). It follows
that \( X \) must also be a permutation matrix. \( \square \)

We can use the cycle-decomposition theorem to get the following result.

**Theorem 2.3** Let \( X_\sigma \) and \( X_\pi \) be vertices of \( B_n \) corresponding to \( \sigma, \pi \in S_n \). The
line segment between \( X_\sigma \) and \( X_\pi \) is an edge if and only if \( \sigma^{-1}\pi \) is a cycle.

**Proof:** This follows directly from the cycle decomposition theorem, Theorem 1.3. \( \square \)

For example, when \( n = 4 \), the points connected to \( X_{(1)} \) are \( X_\sigma \) for \( \sigma \) being any pure
cycle except \((1)\) itself: \((1234),(1324),(12),(34),(13),(24),(14),(23),(123),(234),
(132),(142),(1243),(143),(1342),(243),(134),(124), \) and \((1423)\). This
implies that there are 20 edges containing any given vertex.

### 2.2 The Permutohedron

Now define the **permutohedron** \( P_n \subset \mathbb{R}^n \) to be the convex hull of all permutations
of the vector \((1,2,\ldots,n)\). Specifically, in \( \mathbb{R}^4 \), we would have the vector \((1,2,3,4)\).
2.2. **THE PERMUTAHEDRON**

Using the symmetric group, we find all $4!=24$ permutations of this vector:

\[
(1,2,3,4) \quad (2,1,3,4) \quad (3,1,2,4) \quad (4,1,2,3)
(1,2,4,3) \quad (2,1,4,3) \quad (3,1,4,2) \quad (4,1,3,2)
(1,3,2,4) \quad (2,3,1,4) \quad (3,2,1,4) \quad (4,2,1,3)
(1,3,4,2) \quad (2,3,4,1) \quad (3,2,1,4) \quad (4,2,3,1)
(1,4,2,3) \quad (2,4,1,3) \quad (3,4,1,2) \quad (4,3,1,2)
(1,4,3,2) \quad (2,4,3,1) \quad (3,4,2,1) \quad (4,3,2,1)
\]

It turns out that if we plot all of these points, we find that $P_4$ lies on a three dimensional hyperplane. This shape can be visualized by first imagining an Egyptian pyramid at the edge of a calm lake. Looking at the pyramid and its reflection as a single object, we get the octahedron. Now, imagine this octahedron enclosed in a cube just too small for it. Thus, the corners of the octahedron are cut off, leaving square faces near where the vertices of the octahedron used to belong. The faces of the octahedron which used to be triangles are now hexagons. Imagine, if you will, the Birkhoff polytope, sitting in 16-dimensional space. Remember that this polytope is made up of all of the matrices of the symmetric group of order four, each matrix being a vertex of this greater polytope. You probably cannot visualize this object, since we have a hard time thinking of objects in more than three dimensions. However, we can see its shadow. As a hand casts a shadow on a wall, the Birkhoff polytope casts a shadow on a three dimensional hyperplane, and that shadow is the permutahedron. This mathematically crude description will now be refined.

### 2.2.1 Face Description

More generally, let $a = (a_1, ..., a_n) \in \mathbb{R}^n$ and define the permutahedron to be:

\[
P_n = \text{conv} \{ x_\sigma \mid \sigma \in S_n \}
\]
Figure 2.1: This is $P_4(1, 2, 3, 4)$, the truncated octahedron.
where
\[ x_\sigma := \begin{pmatrix} a_{\sigma(1)} \\ \vdots \\ a_{\sigma(n)} \end{pmatrix} = X_\sigma \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \]
for \( \sigma \in S_n \).

**Theorem 2.4** If the coordinates of \( a \) are pairwise distinct, then the dimension of \( P_n(a) \) is \( n - 1 \).

**Proof:** Since \( P_n(a) \) is contained in the hyperplane with equation \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i \), its dimension is at most \( n - 1 \). To see that the dimension of \( P_n(a) \) is equal to \( n - 1 \), check that \( (a_1, ..., a_n) \) and the \( n - 1 \) points obtained by transposing \( a_i, a_{i+1} \) for \( i = 1, ..., n - 1 \) are affinely independent. \( \square \)

Our next goal is Theorem 2.7, finding an inequality description for \( P_n(a) \). To prove this description of \( P_n(a) \), we must first learn some things about majorizing vectors.

**Definition 2.5** The vector \( x = (x_1, ..., x_n) \) majorizes the vector \( y = (y_1, ..., y_n) \), written \( x \succ y \) (we use altered notation from that used by other sources for simplicity) if, after reordering when necessary, \( x_1 \leq ... \leq x_n \) and \( y_1 \leq ... \leq y_n \), and
\[ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_k \]
and
\[ \sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \quad \text{for } k = 1, ..., n - 1. \]

The following lemma, due to Schur, gives the necessary and sufficient conditions for the majorization of vectors. Recall that a doubly stochastic matrix is exactly an element of the Birkhoff polytope, i.e., an \( n \times n \) matrix with nonnegative entries whose row and column sums are 1.

**Lemma 2.6** The vector \( x \) majorizes the vector \( y \) if and only if there is a doubly stochastic matrix \( \Delta \) such that \( x = \Delta y \).
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PROOF: Suppose $x_1 \leq \ldots \leq x_n$ and $y_1 \leq \ldots \leq y_n$. The proof goes by induction.
Suppose $\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$ for $k = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. It follows that
$\sum_{i=1}^{n} x_i = x_n + \sum_{i=1}^{n-1} x_i = y_n + \sum_{i=1}^{n-1} y_i$. This implies that $y_n - x_n = \sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} y_i \geq 0$. Therefore $y_n \geq x_n$. Then $y_n \geq x_n \geq x_1 \geq y_1$ which implies that there exists some $k$ such that $y_{k+1} \geq x_n \geq y_k$. For this $k$, choose $0 \leq \lambda \leq 1$ such that
$x_n = \lambda y_k + (1-\lambda) y_{k+1}$.

Define $\bar{x} = (x_1, \ldots, x_{n-1})$, and $\bar{y} = (y_1, y_2, \ldots, y_{k-1}, (1-\lambda)y_k + \lambda y_{k+1}, y_{k+2}, \ldots, y_n)$,
where $0 \leq \lambda \leq 1$. Note that the components of $\bar{y}$ are in increasing order of magnitude, i.e., $\bar{y}_1 \leq \bar{y}_2 \leq \ldots \leq \bar{y}_{n-1}$. To see this, we just need to check

$$y_{k-1} \leq (1-\lambda)y_k + \lambda y_{k+1} \leq y_{k+2}$$

Since $x_n = \lambda y_k + (1-\lambda) y_{k+1}$ we have:

$$\bar{y}_k = (1-\lambda)y_k + \lambda y_{k+1} = y_k + y_{k+1} - \lambda y_k - (1-\lambda)y_{k+1} = y_k + y_{k+1} - x_n.$$ 

Since $y_{k+1} \geq x_n$, it follows that

$$y_{k-1} \leq y_k \leq y_k + (y_{k+1} - x_n) = \bar{y}_k$$

Also, since $y_k \leq x_n$, it follows that

$$\bar{y}_k = y_{k+1} + (y_k - x_n) \leq y_{k+1} \leq y_{k+2}.$$ 

Thus, we have shown that the components of $\bar{y}$ are increasing.

Now we want to show that $\bar{x} \succ \bar{y}$. Calculate:

$$\sum_{i=1}^{n-1} \bar{y}_i = \sum_{i=1}^{n-1} \bar{y}_i = y_1 + \cdots + y_{k-1} + \bar{y}_k + y_{k+1} + \cdots + y_n = y_1 + \cdots + y_{k-1} + (y_k + y_{k+1} - x_n) + y_{k+2} + \cdots + y_n = y_1 + \cdots + y_n - x_n = x_1 + \cdots + x_n - x_n = \sum_{i=1}^{n-1} \bar{x}_i$$
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Secondly, if \( \ell < k \), we have

\[
\sum_{i=1}^{\ell} \tilde{x}_i = \sum_{i=1}^{\ell} x_i \geq \sum_{i=1}^{\ell} y_i = \sum_{i=1}^{\ell} \tilde{y}_i.
\]

If \( \ell \geq k \), since \( x_n \geq x_{t}, x \succ y \), and \( \tilde{y}_k = y_k + y_{k+1} - x_n \), we have \( x_1 + \cdots + x_{t+1} \geq y_1 + \cdots + y_{t+1} \) which implies

\[
\sum_{i=1}^{\ell} \tilde{x}_i = \sum_{i=1}^{\ell} x_i \geq y_1 + \cdots + y_{k-1} + (y_k + y_{k+1} - x_n) + y_{k+2} + \cdots + y_{t+1} = \sum_{i=1}^{\ell} \tilde{y}_i.
\]

Hence, \( \tilde{x} \succ \tilde{y} \).

By the inductive hypothesis, there is a \( (n-1) \times (n-1) \) doubly stochastic matrix \( \Delta \) such that \( \tilde{x} = \Delta \tilde{y} \). In full form, this is:

\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_{n-1}
\end{pmatrix}
= \begin{pmatrix}
\delta_{1,1} & \cdots & \delta_{1,n-1} \\
\vdots & \ddots & \vdots \\
\delta_{n-1,1} & \cdots & \delta_{n-1,n-1}
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_{k-1} \\
(1 - \lambda)y_k + \lambda y_{k+1} \\
y_{k+2} \\
\vdots \\
y_n
\end{pmatrix}
\]

In the \( n \times n \) case, we want to find the matrix which relates \( x \) and \( y \). We find this matrix by splitting the \( k^{th} \) column of \( \Delta = (\delta_{ij}) \) into two and adding a final row:

\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_{n-1} \\
x_n
\end{pmatrix}
= \begin{pmatrix}
\delta_{1,1} & \cdots & (1 - \lambda)\delta_{1,k} & \lambda \delta_{1,k} & \cdots & \delta_{1,n-1} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\delta_{n-1,1} & \cdots & (1 - \lambda)\delta_{n-1,k} & \lambda \delta_{n-1,k} & \cdots & \delta_{n-1,n-1}
0 & \cdots & 0 & (1 - \lambda) & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_{k-1} \\
(1 - \lambda)y_k + \lambda y_{k+1} \\
y_{k+2} \\
\vdots \\
y_n
\end{pmatrix}
\]

This new matrix is doubly stochastic. \( \Box \)

We now have the tools necessary for proving the following theorem:

**Theorem 2.7** The permutahedron generated by the vector \( a = (a_1, \ldots, a_n) \), with \( a_1 \leq \cdots \leq a_n \), denoted \( P_n(a) \), has the inequality description

\[
P_n(a) = \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, x(S) \geq \alpha_S \text{ for all } S \subset [n] \},
\]
where \( x(S) = \sum_{i \in S} x_i, \alpha_S = \sum_{i=1}^{[S]} a_i, \) and \([n] = \{1, 2, \ldots, n\} \).

**Proof:** Given \( x \in \mathbb{R}^n \), let \( \tilde{x} \) be a point in \( \mathbb{R}^n \) obtained by permuting the coordinates of \( x \) so that they appear in non-decreasing order. It follows that for any \( S \subseteq [n] \), we have \( x(S) \geq \sum_{i=1}^{[S]} \tilde{x}_i \). Using this fact, one can see that the above inequalities describe the set of all points \( x \) which majorize \( a \). Using the previous lemma, this is the set of all points of the form \( x = \Delta y \) as \( \Delta \) runs over all of the doubly-stochastic matrices; that is, all points in \( B_n \). This is the projection definition of the permutahedron. Therefore, the above inequalities yield the permutahedron \( P_n(a) \). \( \square \)

Now that we know the inequality description of \( P_n(a) \), we can go on to describe the lattice of faces of \( P_n(a) \), and specifically we can determine information about the facets, vertices, and edges of \( P_n(a) \). We will use the following result.

**Lemma 2.8** Let \( S, T \) be sets, and let \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) with \( a_1 \leq \cdots \leq a_n \).

Define the function \( \alpha_S := \sum_{i=1}^{[S]} a_i \). Then \( \alpha_S + \alpha_T \leq \alpha_{S \cap T} + \alpha_{S \cup T} \). If \( a_1 < \cdots < a_n \), then the inequality becomes equality if and only if \( S \subseteq T \) or \( T \subseteq S \).

**Proof:** Define \( |S \cap T| = u, |S| = u + v, |T| = u + w \). Then \( |S \cup T| = u + v + w \). It follows that
\[
\alpha_S = a_1 + \cdots + a_{u+v} \\
\alpha_T = a_1 + \cdots + a_{u+w} \\
\alpha_{S \cap T} = a_1 + \cdots + a_{u+v+w} \\
\alpha_{S \cup T} = a_1 + \cdots + a_u
\]

Furthermore,
\[
(\alpha_{S \cup T} + \alpha_{S \cap T}) - (\alpha_S + \alpha_T) = (\alpha_{S \cup T} - \alpha_S) - (\alpha_T - \alpha_{S \cap T}) \\
= (a_{u+v+1} + \cdots + a_{u+v+w}) - (a_{u+1} + \cdots + a_{u+w}) \geq 0.
\]

If \( a_1 < \cdots < a_n \) note that the last equation is equal to zero if and only if \( v = 0 \), that is, when \( S \subseteq T \), or when \( w = 0 \), giving the trivial result of \( 0 = 0 \) when \( T \subseteq S \). \( \square \)

**Corollary 2.9 [Billera]** Let \( a_1 < \cdots < a_n \). \( F \) is a face of \( P_n(a) \) of codimension \( k \) if and only if equality in \( x(S) \geq \alpha_S \) holds for precisely \( k \) distinct proper subsets lying in a chain \( S_1 \subset \cdots \subset S_k \subset [n] \).
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PROOF If \( x \in P_n(a) \) satisfies \( x(S) = \alpha_S \) and \( x(T) = \alpha_T \) then

\[
\alpha_S + \alpha_T = x(S) + x(T) = x(S \cup T) + x(S \cap T) \geq \alpha_{S \cup T} + \alpha_{S \cap T}.
\]

It follows from the previous lemma that \( \alpha_S + \alpha_T = \alpha_{S \cup T} + \alpha_{S \cap T} \) and further that \( S \subset T \) or \( T \subset S \). Thus, equality holds in \( x(S) \geq \alpha_S \) for \( k \) proper subsets \( S \) if and only if the subsets form a chain \( S_1 \subset \cdots \subset S_k \subset [n] \). The resulting system of linear equations will look something like \( x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_k \) which are necessarily linearly independent. \( \Box \)

**Note:** From now on, we will assume \( a_1 < a_2 < \cdots < a_n \).

Thus, with the above assumption, the face lattice of \( P_n(a) \) is the same as as the lattice of chains of subsets in \([n]\), ordered by refinement. For an alternative description, denote by \( \Pi_n \) the partially ordered set of all ordered partitions of \([n]\), ordered by refinement. The elements of \( \Pi_n \) are ordered tuples \( \pi = (Q_1, \ldots, Q_k) \) where the \( Q_i \) are pairwise disjoint subsets of \([n]\) whose union is \([n]\). Elements smaller than \( \pi \) have the form \((Q_1, \ldots, Q_{ij}, \ldots, Q_k, \ldots, Q_{jk})\) where \((Q_{ij}, \ldots, Q_{jk})\) is an ordered partition on \( Q_i \). For example, in \( \Pi_4 \), \( ([1, 4], [3], [2]) \leq ([1, 3, 4], [2]) \). If we include in \( \Pi_n \) an element \( \hat{0} \) such that \( \hat{0} \leq \pi \) for every ordered partition \( \pi \), then \( \Pi_n \) forms a lattice. Look at a sample lattice, \( \Pi_3 \):

\[
\begin{array}{ccc}
\{1, 2, 3\} & \searrow & \{1, 3\}, \{2\} \\
\downarrow & & \downarrow \\
\{1, 2\}, \{3\} & \searrow & \{1\}, \{2, 3\} \\
\downarrow & & \downarrow \\
\{1\}, \{2\}, \{3\} & \searrow & \hat{0}
\end{array}
\]

where each layer is less than the layer above it.

**Proposition 2.10** \( P_n(a) \) is isomorphic to \( \Pi_n \).
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PROOF: Define a mapping \( \tau : P_n(a) \to \Pi_n \) as follows: For a face \( F \subset P_n(a) \) of codimension \( k \), let \( S_1 \subset \cdots \subset S_k \subset [n] \) be the chain given in Corollary 2.9. For notational purposes, let \( S_0 = \emptyset \) and \( S_{k+1} = [n] \). Then define \( \tau(F) = (Q_1, \ldots, Q_{k+1}) \) where \( Q_i := S_i \setminus S_{i-1} \). It is straightforward to check that \( \tau : P_n(a) \leftrightarrow \Pi_n \) is an isomorphism of lattices which sends a face of codimension \( k \) to a \((k + 1)\)-tuple in \( \Pi_n \). \( \square \)

Corollary 2.11 [Billera] Faces of \( P_n(a) \) are combinatorially equivalent to \( P_{n_1}(a) \times \cdots \times P_{n_k}(a) \) where \( n_1 + \cdots + n_k = n \).

PROOF: Under the isomorphism defined in the previous proposition, the face lattice of a face of \( P_n(a) \) is isomorphic to an interval \([\hat{0}, \pi]\) in \( \Pi_n \) where \( \pi = (Q_1, \ldots, Q_k) \), using the notations from the proof of the proposition. Letting \( n_i = |Q_i| \), it is easy to see that the interval \([\hat{0}, \pi]\) is isomorphic as a lattice to \( \Pi_{n_1}(a) \times \cdots \times \Pi_{n_k}(a) \). \( \square \)

Let \( f = (f_0, \ldots, f_{n-1}) \in \mathbb{Z}^n \) where \( f_i \) is the number of faces of \( P_n(a) \) of dimension \( i \). This is called the \( f \)-vector of \( P_n(a) \).

Theorem 2.12 [YKK] The components of the \( f \)-vector of the permutation polytope \( P_n(a) \) are given by, for all \( k \in [n - 1] \),

\[
f_k(P_n(a)) = \sum \frac{n!}{t_1!t_2! \cdots t_{n-k}!}
\]

where the sum is carried out over all positive integral solutions of the equation \( t_1 + t_2 + \cdots + t_{n-k} = n \).

PROOF: According to Corollary 2.9, faces of \( P_n(a) \) of dimension \( k \) have a one-to-one correspondence with ordered partitions \((Q_1, \ldots, Q_{n-k})\). So \( f_k \) is given by the number of \((n - k)\)-tuples \((Q_1, \ldots, Q_{n-k})\) where the \( Q_i \) are disjoint, non-empty, and \( Q_1 \cup \cdots \cup Q_{n-k} = [n] \). The result follows from standard combinatorial analysis. \( \square \)

For an example, calculate the \( f \)-vector for \( P_4(a) \).

Take \( n = 4 \).

For \( k = 0 \), we write \( 1 + 1 + 1 + 1 = 4 \) to get \( f_0 = 4! = 24 \).
For $k = 1$, we write $2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 4$ to give

$$f_1 = \frac{4!}{2!1!1!} + \frac{4!}{1!2!1!} + \frac{4!}{1!1!2!} = 36.$$ 

For $k = 2$, we write $1 + 3 = 2 + 2 = 3 + 1 = 4$ to give

$$f_2 = \frac{4!}{1!3!} + \frac{4!}{2!2!} + \frac{4!}{3!1!} = 14.$$ 

For $k = 3$, we write $4=4$ to give

$$f_3 = \frac{4!}{4!} = 1$$

Therefore, the $f$-vector is

$$f(P_n(a)) = (24, 36, 14, 1).$$

We would now like to give an explicit description of the vertices of each face of $P_n(a)$. Denote the $k$-face corresponding to the ordered partition $(Q_1, ..., Q_{n-k})$ by $F(Q_1, ..., Q_{n-k})$. Let $\Phi(Q_1, ..., Q_{n-k}) = \{\sigma \mid \sigma(\bigcup_{i=1}^{m} Q_i) = \{1, 2, ..., \bigcup_{i=1}^{m} Q_i\}\}$. 

**Theorem 2.13 ([YKK])** The vertices of the $k$-face $F(Q_1, ..., Q_{n-k})$ are the points $x_\sigma$ for all $\sigma \in \Phi(Q_1, ..., Q_{n-k})$.

**Proof:** We will first show that the vertex $x_\sigma$ for $\sigma \in \Phi(Q_1, ..., Q_{n-k})$, lies in the face $F(Q_1, ..., Q_{n-k})$. For each $m \in [n-k]$, let $S_m := \bigcup_{i=1}^{m} Q_i$, then $\sigma(S_m) = \{1, 2, ..., |S_m|\}$. Therefore, $x_\sigma(S_m) = \sum_{i \in S_m} x_{\sigma(i)} = \sum_{i \in S_m} a_{\sigma(i)} = \sum_{i=1}^{|S_m|} a_i = \alpha_{S_m}$, as required.

On the other hand, if $\sigma \notin \Phi(Q_1, ..., Q_{n-k})$, choose an $m \in [n-k]$ such that $\sigma(S_m) \neq \{1, 2, ..., |S_m|\}$. We have $x_\sigma(S_m) = \sum_{i \in S_m} x_{\sigma(i)} = \sum_{i \in S_m} a_{\sigma(i)} > \sum_{i=1}^{|S_m|} a_i = \alpha_{S_m}$. The last inequality follows since $a_1 < a_2 < ... < a_m$. Since $x_\sigma(S_m) \neq \alpha_{S_m}$, we have $x_\sigma \notin F(Q_1, ..., Q_{n-k})$. \(\Box\)

**Theorem 2.14** ([YKK]) The vertices of $P_n(a)$ adjacent to the vertex $x_\pi$ are the vertices obtained by transposing some pair of adjacent components of $x_\pi$. 
Proof: Suppose the line segment between $x_{\pi}$ and $x_{\sigma}$ forms an edge for some $\sigma \in S_n$. The edge then has the form $F(Q_1, \ldots, Q_{n-1})$. Since $\bigcup Q_i = [n]$ and the $Q_i$ are pairwise disjoint, it follows that each $Q_i$ except for exactly one, say $Q_k$, has one element and $Q_k$ has two elements. Say $Q_1 = \{q_1\}, \ldots, Q_{k-1} = \{q_{k-1}\}, Q_k = \{q_k, q_{k+1}\}, Q_{k+1} = \{q_{k+2}\}, \ldots, Q_{n-1} = \{q_n\}$. Then $\Phi(Q_1, \ldots, Q_{n-1})$ has two elements, $\sigma$ and say $\tau$. We have $\sigma(i) = \tau(i) = q_k$ for $i = 1, \ldots, k-1$ and $\sigma(i) = \tau(i) = q_{k-1}$ for $i = k+2, \ldots, n-1$. Without loss of generality, we can take $\sigma(k) = \tau(k+1) = q_k$ and $\sigma(k+1) = \tau(k) = q_{k+1}$. Thus $x_{\sigma,i} = a_{\sigma(i)} = x_{\tau,i}$ for $i \neq k, k+1$ and $x_{\sigma,k} = x_{\tau,k+1} = a_{q_k}$ and $x_{\sigma,k+1} = x_{\tau,k} = a_{q_{k+1}}$. Thus, we get a transposition in two adjacent places. $\square$
Chapter 3

The Alternating Group

3.1 The Alternating Polytope

The convex hull in $\mathbb{R}^{n^2}$ of the set of even permutation matrices forms the alternating polytope $E_n$.

**Theorem 3.1 (Brualdi)** $E_n$ is an $(n-1)^2$ dimensional polytope with $n!/2$ vertices.

**Sketch of Proof:** We know that each $X_\sigma$ for $\sigma \in A_n$ is a vertex by Theorem 1.2. There are $n!/2$ elements of the alternating group $A_n$, so $E_n$ has $n!/2$ vertices.

The proof that $E_n$ is $(n-1)^2$ dimensional involves showing that there exist $(n-1)^2$ even permutation matrices $P_0 = I_n, P_1, \ldots, P_{(n-1)^2}$ such that the set of matrices $\{P_i - P_0 \mid 1 \leq (n-1)^2\}$ is linearly independent. Please see [Brualdi] for the complete proof. \(\square\)

The facet defining equations and the combinatorial structure of $E_n$ are not known in general. However, we have the following description of the edges of $E_n$. We can use the cycle-decomposition theorem again to describe the edges of $E_n$. It tells us when the line between two vertices $X_\sigma$ and $X_\pi$ is an edge.

**Theorem 3.2 (Brualdi)** Let $\sigma$ and $\pi$ be distinct permutations in $A_n$. Then the line $\{X_\sigma, X_\pi\}$ is an edge of $E_n$ if and only if the cycle decomposition of $\sigma^{-1}\pi$ consists
of exactly 1 cycle of odd length, or exactly two cycles of even length.

PROOF: The line segment \( \{X_\sigma, X_\pi\} \) is an edge of \( E_n \) if and only if \( \sigma^{-1}\pi \) cannot be decomposed into two nontrivial disjoint elements of \( A_n \), by the cycle decomposition theorem, (Theorem 1.3). This can only occur when \( \sigma^{-1}\pi \) is one odd length cycle or the product of two even length cycles. \( \square \)

### 3.2 The Alternahedron

The convex hull of all even permutations of the point \( a = (a_1, ..., a_n) \), where the coordinates are pairwise distinct, is defined to be the **alternahedron**, denoted \( H_n(a) \), as discussed in the first chapter. Define \( O_n \) to be the set of odd permutations: 

\[ O_n = \{ \phi \mid \phi \in S_n \setminus A_n \}. \]

By Theorem 2.14, we know that for \( \phi \in O_n \), the \( n - 1 \) affinely independent vertices adjacent to \( a_\phi \), call \( a_\delta_i \) for \( i = 1, ..., n - 1 \), are all even. Then the unique hyperplane \( T_\phi \) which passes through all the vertices \( a_\delta_i \) strictly separates \( a_\phi \) from the polytope \( \text{conv} \{a_\tau \mid \tau \in S_n \setminus \phi\} \) which it supports. Thus, the intersection of the polytope \( P_n(a) \) and all the half spaces \( T_\phi, \sigma \in O_n \) is precisely the polytope \( H_n(a) \). The equations for these half spaces are determined in [YKK]. Setting

\[ c_1 = a_1, c_2 = a_2, c_i = c_{i-1} - \frac{(a_{n-i+1} - a_n)(a_1 - a_2)}{a_{n-i+1} - a_{n-i+2}} \]

The desired hyperplane \( T_\sigma \) is given by the equation

\[ \sum_{i=1}^{n} c_{\sigma(i)}x_i = \sum_{i=1}^{n} c_{i}a_{n-i+1} + (a_{n-1} - a_n)(a_1 - a_2). \]

**Theorem 3.3 ([YKK], Theorem 3.13)** The even permutation polytope \( H_n(a) \) is given by the inequalities of the permutation polytope

\[ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i, \quad x(S) \geq a_s \quad \text{for all } S \subset [n] \]
and the halfspaces
\[ \sum_{i=1}^{n} c_{\phi(i)} x_i \geq \sum_{i=1}^{n} c_i a_{n-i+1} + (a_{n-1} - a_n)(a_1 - a_2) \quad \forall \phi \in O_n \]
If \( n > 4 \), then every inequality defines a face.

Since we can think of the alternahedron as the polytope constructed by cutting
off half of the vertices of the permutahedron, and we know the permutahedron has
\( n!/2 \) vertices, we know that the alternahedron has \( n!/2 \) vertices.

**Theorem 3.4** The alternahedron \( H_n(a) \) has dimension \( n - 1 \).

**Proof:** The points of \( H_n(a) \) adjacent to an odd vertex of the full permutahedron
are affinely independent. \( \Box \)

There are many unsolved mysteries concerning the alternahedron. We have
experimental data for the first few cases; however, the alternahedron grows large
very quickly, and any information above \( n = 6 \) takes a very long time for the
computer to compile. We have the following conjecture about the alternahedron:

**Conjecture 3.5** The alternahedron \( H_n(a) \) has \( n!/2 \) facets containing \( n-1 \) vertices.
This says that \( H_n(a) \) has \( n!/2 \) simplicial facets.

For \( n = 3 \) and 4, these are all of the facets of the alternahedron; however,
for \( n > 4 \), \( H_n(a) \) has other facets with varying numbers of vertices on them. For
example, \( H_5(a) \) has 60 facets with 4 vertices, which are the facets discussed in the
conjecture above. However, it also has 10 facets with 12 vertices, and 20 facets with
6 vertices. We also have information on \( H_6(a) \), which has 360 facets with 5 vertices
(these are the \( n!/2 \) simplicial facets again), as well as 20 facets with 18 vertices, 30
facets with 24 vertices, and 12 facets with 60 vertices. We have not been able to
predict in general how many facets we will have, nor how many vertices are on each
facet. Since we have been unable to find general theorems for the alternahedron, let
us look at some examples in detail.
Using the computer program PORTA (see [PORTA]), we collected the following data for the alternahedron:

<table>
<thead>
<tr>
<th>$n$</th>
<th>number of vertices</th>
<th>dimension</th>
<th>vertices on facets</th>
<th>number of facets</th>
<th>number of facets each vertex is on</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>3</td>
<td>3</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>4</td>
<td>4,6,12</td>
<td>90</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>360</td>
<td>5</td>
<td>5,18,24,60</td>
<td>422</td>
<td>10</td>
</tr>
</tbody>
</table>

Each generic $H_4(a)$ is combinatorially equivalent to an icosahedron. Checking by hand, we noticed that any copy of $A_4$ sitting inside $A_5$ obtained by fixing one element produced a facet of $H_5(a)$. It would be interesting to completely determine the correspondence between $H_5(a)$ and subgroups of $A_5$.

This information leads us to ask if there are formulae predicting the following data:

- Number of vertices on facets
- Number of facets
- Number of facets each vertex is on
3.2. THE ALTERNAHEDRON

Figure 3.1: This is $H_4(1, 2, 3, 4)$, which is combinatorially equivalent to the icosahedron.
CHAPTER 3. THE ALTERNATING GROUP
Chapter 4

Dihedral group

4.1 The Dihedral Polytope

The convex hull of the permutation matrices of the dihedral group forms the dihedral polytope $T_n$ with $2n$ vertices, (Theorem 1.2).

Theorem 4.1 The dimension of $T_n$ is $2n - 3$ when $n$ is even, and $2n - 2$ when $n$ is odd.

Proof: We need to find the dimension of the linear space of hyperplanes containing $T_n$. To do this, form a matrix whose rows are the elements of the dihedral group, thought of in the usual way as points of $\mathbb{R}^n$. Augment the matrix by adding a final column of 1s (in order to account for the constants in the equations for the hyperplanes). Call the resulting $2n \times (n^2 + 1)$ matrix $Z$. We will show that $Z$ has rank $2n - 2$ or $2n - 1$ depending on whether $n$ is even or odd. Since elements of the kernel of $Z$ correspond exactly with hyperplanes containing $T_n$, we have

$$\dim T_n = n^2 - \dim \ker(T_n) = n^2 - (n^2 + 1 - \text{rank}(Z)) = \text{rank}(Z) - 1,$$

as required.

To explicitly construct $Z$, take permutations generating the dihedral group: pick $\rho = (1,2,\ldots,n)$ for the rotation, and choose the reflection fixing 1, namely...
\( \phi = (2, n)(3, n - 1) \cdots \). The first \( n \) rows of \( Z \) will correspond the rotations:

\[
(1), \rho, \rho^2, \ldots, \rho^{n-1},
\]
in the order listed, and the last \( n \) rows will correspond to the reflections

\[
\phi, \phi \rho, \phi \rho^2, \ldots, \phi \rho^{n-1},
\]
in the order listed. Now augment with a final column of 1s. Letting \( e_i \) denote the \( i \)-th standard basis vector for \( \mathbb{R}^n \) allows us to write the resulting matrix as:

\[
Z = \begin{pmatrix}
e_1 & e_2 & e_3 & \cdots & e_n & 1 \\
e_n & e_1 & e_2 & \cdots & e_{n-1} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e_2 & e_3 & e_4 & \cdots & e_1 & 1 \\
e_1 & e_n & e_{n-1} & \cdots & e_2 & 1 \\
e_n & e_{n-1} & e_{n-2} & \cdots & e_1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e_2 & e_1 & e_n & \cdots & e_3 & 1
\end{pmatrix}
\]

Hence, with this notation, each column except the last represents \( n \) columns of \( Z \). Subtracting row \( i \) from row \( n + i \) for \( i = 1, \ldots, n \) gives

\[
\begin{pmatrix}
e_1 & e_2 & e_3 & \cdots & e_n & 1 \\
e_n & e_1 & e_2 & \cdots & e_{n-1} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e_2 & e_3 & e_4 & \cdots & e_1 & 1 \\
\bar{e}_1 - e_2 & e_{n-1} - e_3 & \cdots & e_2 - e_n & 0 \\
\bar{e}_n - e_1 & e_{n-2} - e_2 & \cdots & e_1 - e_{n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{e}_1 - e_3 & e_n - e_4 & \cdots & e_3 - e_1 & 0
\end{pmatrix}
\]

Noting the shape of the first and last column, it suffices to show that the following
submatrix has rank $n - 2$ or $n - 1$ depending on whether $n$ is even or odd:

$$Z' = \begin{pmatrix}
  e_n - e_2 & e_{n-1} - e_3 & \cdots & e_2 - e_n \\
  e_{n-1} - e_1 & e_{n-2} - e_2 & \cdots & e_1 - e_{n-1} \\
  \vdots & \vdots & & \vdots \\
  e_1 - e_3 & e_n - e_4 & \cdots & e_3 - e_1
\end{pmatrix}$$

First we treat the even case. It is easy to check that the sum of the even-numbered rows and the sum of the odd-numbered rows are both zero. For instance, consider the sum of the odd-numbered rows of the first column of $Z'$:

$$(e_n - e_2) + (e_{n-2} - e_n) + (e_{n-4} - e_{n-2}) + \cdots + (e_4 - e_6) + (e_2 - e_4) = 0.$$ 

Summing up the odd-numbered rows of each column produces a similar telescoping sum. The same argument works for the sum of the even-numbered rows. Thus, we have shown that the rank of $Z'$ is at most $n - 2$. To finish the argument in the even case, note that the following vectors from the first column of $Z'$ are obviously linearly independent: $e_1 - e_3, e_2 - e_4, \ldots, e_{n-2} - e_n$.

We now treat the odd case. Here, it is easy to check, as in the even case, that the sum of all of the rows is zero. Hence, the rank of $Z'$ is at most $n - 1$. Again, the vectors $v_1 := e_1 - e_3, v_2 := e_2 - e_4, \ldots, v_{n-2} := e_{n-2} - e_n$ from the first column of $Z'$ are clearly linearly independent. The vector $e_{n-1} - e_1$ also occurs in the first column. Adding the odd-numbered $v_i$’s to $e_{n-1} - e_1$ produces the vector $v_{n-1} = e_{n-1} - e_n$. The vectors $v_1, \ldots, v_{n-1}$ are clearly linearly independent, hence, the rank of $Z'$ in the odd case is $n - 1$.  

We can again use the cycle-decomposition theorem that we used in the previous chapters to describe the edges of $T_n$.

**Theorem 4.2** Every vertex of $T_n$ with $n > 4$ is connected to every other vertex of $T_n$ by an edge of $T_n$. 
CHAPTER 4. DIHEDRAL GROUP

Proof: Without loss of generality, examine the line segment between $X_{\pi}$ and $X_{\sigma}$. By the cycle decomposition theorem, (Theorem 1.3), this line segment is an edge if and only if the cycle decomposition of $\pi$ cannot be factored into two parts, both of which form elements of $D_n$. Suppose we could factor $\pi$ in such a way, say $\pi = \pi_1 \pi_2$. So a fixed point in $\pi$ is fixed in $\pi_1 \pi_2$ which implies it is fixed in both $\pi_1$ and $\pi_2$. But, since both $\pi_1$ and $\pi_2$ are elements of $D_n$, at most two elements can be fixed by either of them. Thus, both $\pi_1$ and $\pi_2$ have at least $n - 2$ nonfixed points. By construction, the points not fixed by $\pi_1$ are disjoint from the points not fixed by $\pi_2$. Since there are only $n$ points altogether, we need $(n - 2) + (n - 2) \leq n$, which implies $n \leq 4$. □

Using the language of the proof of Theorem 4.2, the only time we can decompose $\pi$ into two non-trivial parts, both of which are in $D_n$, is the case $D_4$, where $(23)(14)$ is composed of $(23)$ and $(14)$, which are both non-trivial elements of $D_4$. It follows that the line segment $\{X_{(1)}, X_{(23)(14)}\}$ is not an edge of $T_4$. The line segment $\{X_{\sigma}, X_{\pi}\}$ is not an edge of $T_4$ when $\sigma^{-1}\pi$ can be factored to $(23)(14)$. These are the only such line segments. This means that all dihedral polytopes $T_n$ with $n > 4$ have edges connecting every pair of vertices. Thus, there are $\binom{2n}{2}$ edges. □

David Perkinson has a proof for the following theorem:

**Theorem 4.3** The odd dihedral polytopes are simplicial.

Our data suggest the following conjectures, for which the proofs are unknown.

**Conjecture 4.4** Each facet of $T_n$ has $2n - 2$ vertices.

The fact that odd dihedral polytopes are simplicial proves this conjecture for odd $n$; however, the proof for even $n$ is unknown.

**Conjecture 4.5** The odd dihedral polytopes have $n^2$ facets. The even dihedral polytopes have $n^2/2$ facets.
Conjecture 4.6  The vertices of the odd dihedral polytopes are on \( n(n-1) \) facets. The vertices of the even dihedral polytopes are on \( n(n-1)/2 = \binom{n}{2} \) facets.

These conjectures are based on computer calculations yielding the following charts.

<table>
<thead>
<tr>
<th>( n )</th>
<th>number of points</th>
<th>dimension</th>
<th>vertices per facet</th>
<th>number of facets each vertex is on</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>9</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>12</td>
<td>12</td>
<td>49</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>13</td>
<td>14</td>
<td>32</td>
</tr>
<tr>
<td>9</td>
<td>18</td>
<td>16</td>
<td>16</td>
<td>81</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>17</td>
<td>18</td>
<td>50</td>
</tr>
</tbody>
</table>

### 4.2 The Dihedron

As in previous chapters, we can take the projection of the dihedral polytope to get the dihedron, \( Q_n(a) \).

**Theorem 4.7** The dihedron has dimension \( n - 1 \).

**Proof:** Look at the subset of rotations of \( D_n \). Consider the set of points created by these matrices acting on the vector \( a \). For generic \( a \), the linear space spanned by these points has dimension \( n \) by a standard result about circulant matrices ([Philip], p.75). Hence the smallest affine space containing \( D_n \) has dimension \( n - 1 \). □

Some point \( a \) is generic if there is an open set \( U \) about \( a \) such that \( Q_n(b) \) has the same combinatorial structure as \( Q_n(a) \) for all \( b \in U \). Unlike the cases we examined in previous chapters, it is possible to find generic points \( a \) and \( a' \) such that \( Q_n(a) \) and \( Q_n(a') \) are not combinatorially equivalent. This behavior was first noted in the dihedron by David Perkinson and Douglas Squirrel in 1996. Previously, this type of behavior was noticed for more complicated groups in [Omn]. Depending on the
point $a$ we choose, we can get dramatically different polytopes. Choose polytopes $Q_5(1,2,6,4,3)$ and $Q_5(2,1,6,4,3)$. They both have the same dimension, 4, and number of vertices, 10, but the first has 35 facets with each vertex being on 14 facets, while the second has only 30 facets, with each vertex laying on 12 facets. Here is a chart of the possible number of facets from $Q_5(a)$ through $Q_8(a)$, each possibility coming from a different generic point $a$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>number of facets</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>30,35</td>
</tr>
<tr>
<td>6</td>
<td>20,32</td>
</tr>
<tr>
<td>7</td>
<td>140,154,168,182,196,210</td>
</tr>
<tr>
<td>8</td>
<td>118,150,190,198,222,230,246</td>
</tr>
<tr>
<td>9</td>
<td>612, 630, 675, 693, 738, 747, 756, 765, 774, 783, 810, 819, 828, 837, 846, 864, 873, 891, 900, 909, 918, 927, 936, 945, 954, 963, 972, 981, 990, 999, 1008, 1017, 1026, 1044</td>
</tr>
</tbody>
</table>

Notice that each possibility for the number of facets differs from another by a multiple of $n$. In fact, for odd $n$, each possibility is a multiple of $n$. To get this data, we used a program called orb, created by Douglas Squirrel. The program generates random points $a$, checks if they are generic in the sense defined above, and then outputs the number of facets on $Q_n(a)$. Although we let the program run for some time to collect the data, it is possible that not every possibility for the number of facets appeared, especially as $n$ increased. We predict that we would find every multiple of $n$, within certain bounds, in an infinite data set. It is an interesting question to ask exactly what these bounds are.
Chapter 5

Questions

In the process of writing this thesis, we found many more questions than we started with, and many of them remain unsolved. The reader may find one or more of them worth pursuing in the future. The questions are ordered by the chapter to which each relates.

Chapter Three: The Alternating Group

- Show that the alternahedron $H_n(a)$ has $n!/2$ facets containing $n - 1$ vertices. This says that $H_n(a)$ has $n!/2$ simplicial facets.

- Find the complete combinatorial structure of $H_5(a)$.

- Completely describe $H_5(a)$ using relations between faces and subgroups.

- Find a formula for the number of vertices on each facet of $H_n(a)$.

- Find a formula for the number of facets of $H_n(a)$.

- Find a formula for the number of facets each vertex of $H_n(a)$ is on.

Chapter Four: The Dihedral Group.
• What are the bounds on the possible number of faces one can get from the
dihedron?

• Show each facet of $T_n$ has $2n - 2$ vertices when $n$ is even.

• Show that the odd dihedral polytopes have $n^2$ facets. The even dihedral
  polytopes have $n^2/2$ facets.

• The vertices of the odd dihedral polytopes are on $n(n - 1)$ facets. The vertices
  of the even dihedral polytopes are on $n(n - 1)/2 = \binom{n}{2}$ facets.

• What do dihedral face lattices look like?
Bibliography


[PORTA] Thomas Christof, PORTA- A Polyhedron Representation Transition Algorithm, version 1.2.1, revised by A. Loebel and M. Stoer, available from the ZIB electronic library ELIB via elib@zib-berlin.de or anonymous ftp from elib.zib-berlin.de, dir “/pub/mathprog”


