1 Fields

A field is a set $F$ with two operations, $+: F \times F \to F$ (addition) and $\cdot: F \times F \to F$ (multiplication), satisfying the following axioms:

**F1.** Addition is commutative. For all $x, y \in F$,

$$x + y = y + x.$$  

**F2.** Addition is associative. For all $x, y, z \in F$,

$$(x + y) + z = x + (y + z).$$

**F3.** There is an additive identity. There is an element of $F$, usually denoted $0$, such that for all $x \in F$,

$$x + 0 = x.$$  

**F4.** There are additive inverses. For all $x \in F$, there is an element $y \in F$ such that

$$x + y = 0.$$  

The element $y$ is denoted $-x$. (Subtraction is then defined by $x - y := x + (-y)$ for all $x, y \in F$.)

**F5.** Multiplication is commutative. For all $x, y \in F$,

$$xy = yx.$$  

**F6.** Multiplication is associative. For all $x, y, z \in F$,

$$(xy)z = x(yz).$$  

**F7.** There is a multiplicative identity. There is an element, usually denoted $1$, such that:

(a) $1 \neq 0$, and
(b) $1x = x$ for all $x \in F$.

**F8.** There are multiplicative inverses. For each nonzero $x \in F$, there is a $y \in F$ such that

$$xy = 1.$$  

The element $y$ is denoted $1/x$ or $x^{-1}$. (Division is then defined by $x \div y := xy^{-1}$ for nonzero $y$.)

**F9.** Multiplication distributes over addition. For all $x, y, z \in F$,

$$x(y + z) = xy + xz.$$  

1
2 Order

An ordered field is a field $F$ with a relation, denoted $<$, satisfying

\begin{enumerate}
\item[O1.] (Trichotomy) For all $x, y \in F$, exactly one of the following statements is true:
\[x < y, \quad y < x, \quad x = y.\]
\item[O2.] (Transitivity) The relation $<$ is transitive, i.e., for all $x, y, z \in F$,
\[x < y \text{ and } y < z \implies x < z.\]
\item[O3.] (Additive translation) For all $x, y, z \in F$,
\[x < y \implies x + z < y + z\]
\item[O4.] (Multiplicative translation) For all $x, y, z \in F$,
\[x < y \text{ and } 0 < z \implies xz < yz.\]
\end{enumerate}

Remark: of course, we write $x > y$ if $y < x$, and we write $x \leq y$ if either $x = y$ or $x < y$.

3 Completeness

Let $S$ be a subset of an ordered field $F$. An element $M \in F$ is an upper bound for $S$ if $M \geq s$ for all $s \in S$. An element $M \in F$ is the least upper bound or supremum for $S$ if it is an upper bound and is less than or equal to every upper bound. In this case, we write $M = \text{lub} S$ or $M = \sup S$. Similarly, an element $m \in F$ is a lower bound for $S$ if $m \leq s$ for all $s \in S$. An element $m \in F$ is the greatest lower bound or infimum for $S$ if it is a lower bound and is greater than or equal to every lower bound. In this case, we write $m = \text{glb} S$ or $m = \inf S$.

An ordered field $F$ is complete if every nonempty subset $S \subseteq F$ which has an upper bound has a least upper bound in $F$.

4 The Real Numbers

Theorem. There exists a complete ordered field.

It can also be shown that any two complete ordered fields are isomorphic, i.e., they are the same except for renaming elements. Thus, there is essentially one complete ordered field, and it is the set of real numbers, $\mathbb{R}$. 