Math 111 Definition of the integral

Let $f$ be a bounded function on a closed interval $[a, b]$. (The word “bounded” here means that there is some constant $B$ such that $-B \leq f(x) \leq B$ for all $x \in [a, b]$.) A partition of $[a, b]$ is a set $P = \{t_0, t_1, t_2, \ldots, t_n\}$ with each $t_i \in [a, b]$ and with $t_0 = a$ and $t_n = b$. By convention, we always take $t_0 < t_1 < \cdots < t_n$. In that case the interval $[t_{i-1}, t_i]$ is called the $i$-th subinterval of $P$. The values the function $f$ takes on the $i$-th interval is denoted $f([t_{i-1}, t_i])$:

$$f([t_{i-1}, t_i]) = \{f(x) : t_{i-1} \leq x \leq t_i\}.$$  

This set is called the image of $[t_{i-1}, t_i]$ under $f$. For each $i = 1, \ldots, n$ define

$$m_i = \text{glb} f([t_{i-1}, t_i]),$$

$$M_i = \text{lub} f([t_{i-1}, t_i]),$$

then define the lower and upper sums for $f$ with respect to $P$:

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

The upper and lower integrals for $f$ are:

$$L\int_{a}^{b} f = \text{lub} \{L(f, P) : P \text{ a partition of } [a, b]\},$$

$$U\int_{a}^{b} f = \text{glb} \{U(f, P) : P \text{ a partition of } [a, b]\}.$$

The function $f$ is integrable if

$$L\int_{a}^{b} f = U\int_{a}^{b} f,$$

and in that case, the common value is the integral of $f$ on $[a, b]$, denoted $\int_{a}^{b} f$ or $\int_{a}^{b} f(x) \, dx$. □

**Riemann sum.** Pick real numbers $c_i$ such that $t_{i-1} \leq c_i \leq t_i$ for $i = 1, \ldots, n$. The quantity

$$R = \sum_{i=1}^{n} f(c_i)(t_i - t_{i-1})$$
is called a Riemann sum for \( f \) relative to the partition \( P \). Note that

\[ L(f, P) \leq R \leq U(f, P) \]

for all choices of the \( c_i \). Thus, a Riemann sum estimates the integral of \( f \), provided the integral exists.