Filters

01° Let $X$ be any set. By a filter on $X$, we mean a nonempty family $\mathcal{F}$ of subsets of $X$ which meets the following conditions:

1. $\emptyset \notin \mathcal{F}$
2. $F \in \mathcal{F}, G \in \mathcal{F} \implies F \cap G \in \mathcal{F}$
3. $F \in \mathcal{F}, F \subseteq H \implies H \in \mathcal{F}$

where $F$, $G$, and $H$ are any subsets of $X$.

02° It may happen that a nonempty family $\mathcal{F}_o$ of subsets of $X$ meets conditions (1) and (2) but (perhaps) not (3). In such a case, we introduce the family $\mathcal{F}$ consisting of all subsets $G$ of $X$ such that there is some $F$ in $\mathcal{F}_o$ for which $F \subseteq G$. Obviously, $\mathcal{F}$ is a filter on $X$, as it meets not only conditions (1) and (2) but also (3). We say that $\mathcal{F}_o$ generates $\mathcal{F}$.

03° For instance, we may select a member $\xi$ of $X$, then take $\mathcal{F}_o$ to be the family consisting of the singleton $\{\xi\}$. In such a case, we refer to the filter generated by $\mathcal{F}_o$ as the principal filter on $X$ defined by $\xi$. We denote it by $\mathcal{P}_\xi$.

04° Let $\mathcal{F}$ be a filter on $X$. Let $A$ and $B$ be subsets of $X$ such that $A \cup B \in \mathcal{F}$. We contend that if $B \notin \mathcal{F}$ then there is a filter $\mathcal{G}$ on $X$ such that:

$$\mathcal{F} \cup \{A\} \subseteq \mathcal{G}$$

To prove the contention, we argue as follows. Let us form the family $\mathcal{G}_o$ of subsets of $X$ of the form $F \cap A$, where $F$ runs through $\mathcal{F}$. Obviously, $\mathcal{G}_o$ meets condition (2). Moreover, if there were some $F$ in $\mathcal{F}$ for which $F \cap A = \emptyset$ then $F \cap (A \cup B) = F \cap B$, so that $B$ would be in $\mathcal{F}$, a contradiction. Consequently, $\mathcal{G}_o$ meets condition (1). Now we need only take $\mathcal{G}$ to be the filter generated by $\mathcal{G}_o$.

Maximal Filters

05° Let $\mathcal{F}$ be the family of all filters on $X$. Let us supply $\mathcal{F}$ with a partial ordering, as follows:

$$\mathcal{F}' \preceq \mathcal{F}' \iff \mathcal{F}' \subseteq \mathcal{F}''$$
where $F'$ and $F''$ are any filters on $X$. With respect to the partial ordering on $F$ just defined, we plan to study the *maximal* filters. These are the filters $U$ on $X$ such that, for any filter $F$ on $X$, if $U \subseteq F$ then $U = F$. Very often, one refers to such filters as *ultrafilters*.

06° Obviously, the principal filters on $X$ are maximal with respect to the foregoing partial ordering. We inquire whether there are any others.

07° Let $U$ be an ultrafilter on $X$. With reference to article 04°, we find that, for any subsets $A$ and $B$ of $X$, if $A \cup B \in U$ then $A \in U$ or $B \in U$. We infer that $U$ meets the *finite union condition*, which is to say that, for any finite family $\mathcal{A}$ of subsets of $X$, if:

$$\bigcup A \in U$$

then there is at least one set $A$ in $\mathcal{A}$ such that $A \in U$.

08° In fact, the foregoing condition characterizes ultrafilters. To see that it is so, let us introduce a filter $\mathcal{F}$ on $X$ which meets the finite union condition and let us suppose that $\mathcal{F}$ is not maximal. Accordingly, we may introduce a filter $\mathcal{G}$ on $X$ and a subset $A$ of $X$ such that $\mathcal{F} \subseteq \mathcal{G}$, $A \notin \mathcal{F}$, and $A \in \mathcal{G}$. Now the subset $A$ and its complement $B$ in $X$ yield $A \cup B \in \mathcal{F}$ while $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Consequently, the supposition is untenable. Hence, $\mathcal{F}$ is maximal.

09° By the foregoing discussion, we infer that, for any ultrafilter $U$ on $X$, $U$ is principal iff:

$$\bigcap U \neq \emptyset$$

In fact, for any member $\xi$ of $X$, if:

$$\xi \in \bigcap U$$

then, for any $V$ in $U$, $\{\xi\} \cup (V \setminus \{\xi\}) \in U$, hence, $\{\xi\} \in U$, so that $U = \mathcal{P}_\xi$.

**Existence of Maximal Filters**

10° From this point forward, let us assume that $X$ is infinite.

11° Let $\mathcal{E}$ be the filter on $X$ consisting of all subsets $E$ for which the complement $\overline{E}$ of $E$ in $X$ is finite. In turn, let $\mathcal{F}_0$ be the family of all filters $\mathcal{F}$ on $X$ such that $\mathcal{E} \subseteq \mathcal{F}$.

12° Verify that $\mathcal{E}$ is not maximal.
13° By a chain in \( F_o \), we mean a subfamily \( C \) of \( F_o \) such that, for any filters \( F' \) and \( F'' \) in \( C \), \( F' \preceq F'' \) or \( F'' \preceq F' \). We may say that \( C \) is linearly ordered. For such a family \( C \), we find that:

\[
\mathcal{G} = \bigcup C
\]

is a filter in \( F_o \) and \( \mathcal{G} \) is an upper bound for \( C \), in the sense that, for each filter \( F \) in \( C \), \( F \subseteq \mathcal{G} \).

14° By the foregoing observation, we conclude that every chain in \( F_o \) is bounded. Now the Lemma of Zorn implies that there exist filters \( \mathcal{U} \) in \( F_o \) which are maximal. Obviously, such filters are maximal in \( F \) as well. And they are not principal.

**The Space of Ultrafilters**

15° Let \( X \) be any set. Let \( U \) be the family of all ultrafilters on \( X \). For amusement, let us note that:

\[
U \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))
\]

We intend to supply \( U \) with a topology. The corresponding topological space proves to have remarkable properties.

16° To that end, let \( A \) be any subset of \( X \). Let \( T_A \) be the subset of \( U \) defined as follows:

\[
T_A = \{ U \in U : A \in U \}
\]

These subsets of \( U \) form the base for the topology on \( U \), soon to be defined.

17° Note that \( T_\emptyset = \emptyset \) and \( T_X = U \). Verify that:

\[
B \subseteq C \implies T_B \subseteq T_C
\]

\[
T_{B \cap C} = T_B \cap T_C, \quad T_{B \cup C} = T_B \cup T_C, \quad T_{X \setminus D} = U \setminus T_D
\]

where \( B, C, \) and \( D \) are any subsets of \( X \).

18° In turn, let \( A \) be any subset of \( \mathcal{P}(X) \). Let \( T_A \) be the subset of \( U \) defined as follows:

\[
T_A = \bigcup_{A \in A} T_A
\]

These subsets of \( U \) form the topology on \( U \). They are the open subsets of \( U \). By definition, they are the various unions of families of basic open subsets of \( U \).
Properties

19° Now let us prove that the topological space $U$ is hausdorff, compact, and extremely disconnected.

20° First, hausdorff. Let $U_1$ and $U_2$ be distinct ultrafilters in $U$: $U_1 \neq U_2$

Of course, there must be some subset $A$ of $X$ such that $A \in U_1$ but $A \notin U_2$. Hence, $B = X \setminus A \in U_2$. Consequently:

$U_1 \in T_A, \; U_2 \in T_B, \; T_A \cap T_B = \emptyset$

It follows that $X$ is hausdorff.

21° Second, compact. Let us introduce an open covering of $U$:

$C_A = \{T_A : A \in A\}$

where $A$ is a subset of $\mathcal{P}(\mathcal{P}(X))$. By the definition of covering:

$\bigcup_{A \in A} T_A = U$

We must show that there is a finite subset $F$ of $A$ such that:

$\bigcup_{A \in F} T_A = U$

To that end, let $B$ be the subset of $\mathcal{P}(X)$ defined as follows:

$B = \bigcup A$

Obviously:

$\bigcup_{B \in B} T_B = \bigcup_{A \in A} (\bigcup_{B \in B} T_B) = \bigcup_{A \in A} T_A = U$

Moreover, for any $B$ in $B$, there is some $A$ in $A$ such that $B \in A$, so that:

$T_B \subseteq T_A$

Now we need only show that there is a finite subset $F$ of $B$ such that:

$(\circ) \quad \bigcup_{B \in F} T_B = U$
In effect, we have reduced the context of a general covering of \( U \) by open subsets to the context of a basic covering of \( U \) by basic open subsets.

22° By the finite union condition, condition (\( \circ \)) is equivalent to the following condition:

\[ \bigcup_{B \in \mathcal{F}} B = X \]

Let us suppose that there is no finite subset \( \mathcal{F} \) of \( \mathcal{B} \) such that condition (\( \bullet \)) holds true. It would follow that the family \( \mathcal{C} \) of complements:

\[ \mathcal{C} = \{ C = X \setminus B : B \in \mathcal{B} \} \]

generates a filter on \( X \). Consequently, there would be an ultrafilter \( \mathcal{U} \) on \( X \) which includes \( \mathcal{C} \). It would follow that:

\[ \mathcal{U} \not\subseteq \bigcup_{B \in \mathcal{B}} \mathcal{T}_B \]

a contradiction. So the supposition is untenable. Hence, there is finite subset \( \mathcal{F} \) of \( \mathcal{B} \) such that condition (\( \bullet \)) holds true. The proof is complete.

23* Verify that, for any subset \( D \) of \( X \), \( \mathcal{T}_D \) is not only open but also compact.

24° Third, extremely disconnected. Let \( \mathcal{A} \) be any subset of \( \mathcal{P}(X) \). We will show that there is a subset \( \mathcal{B} \) of \( X \) that:

\[ \text{clo}(\mathcal{T}_\mathcal{A}) = \mathcal{T}_\mathcal{B} \]

In this way, we will prove that the closure of any open subset of \( U \) is itself open, in fact, that it is a basic open subset of \( U \).

25° To that end, let us introduce the following sets:

\[ B = \bigcup \mathcal{A}, \quad C = X \setminus B, \quad \mathcal{B} = \mathcal{P}(B), \quad \mathcal{C} = \mathcal{P}(C) \]

One can easily check that \( \mathcal{T}_\mathcal{C} \) is the largest (under the relation of inclusion) among all open subsets of \( U \) which are disjoint from \( \mathcal{T}_\mathcal{A} \). Consequently:

\[ U \setminus \mathcal{T}_\mathcal{C} = \text{clo}(\mathcal{T}_\mathcal{A}) \]
However, $T_B = T_B$ and $T_C = T_C$. It follows that:

$$T_B = U \setminus T_C = \text{clo}(T_A)$$

The proof is complete.

26. For each member $\xi$ of $X$, we may identify $\xi$ with the corresponding principal ultrafilter $P_\xi$. In this way, we obtain an injective mapping $\pi$ carrying $X$ to $U$:

$$\pi(\xi) = P_\xi$$

where $\xi$ is any member of $X$. Show that the range of $\pi$ is dense in $U$:

$$\text{clo}(\text{ran}(\pi)) = U$$

Show that, for each member $\xi$ of $X$, $\pi(\xi)$ is an isolated point in $U$. In fact:

$$\{P(\xi)\} = T_{\{\xi\}}$$