RIEMANN/RICCI/WEYL

Thomas Wieting
Reed College, 1994

Introduction

1° We plan to explain the following canonical decomposition of the curvature tensor $K$:

$$K = G \bullet \left( \frac{1}{2} R - \frac{1}{12} r G \right) + W$$

In this context, $G$ is the given metric tensor on space-time, $K$ is the riemann curvature tensor defined by $G$, $R$ is the ricci tensor defined by $K$, $r$ is the ricci scalar, and $W$ is the associated weyl tensor. The Kulkarni/Nomizu operator $\bullet$ will be defined in due course.

Various Tensors

2° We must consider tensors of riemann type and tensors of ricci type. The former are tensors $L$ of valence $(0,4)$:

$L_{ijkl}$

meeting the following conditions:

$$L_{ijkl} = -L_{ijlk}$$
$$L_{klij} = L_{ijlk}$$
$$L_{ijlk} = -L_{ijkl}$$

and also the condition:

$$L_{ijkl} + L_{jikl} + L_{kijl} = 0$$

The latter are tensors $S$ of valence $(0,2)$:

$S_{ij}$

meeting the following condition:

$$S_{ji} = S_{ij}$$

Of course, the metric tensor $G$ is itself of ricci type. With regard to condition (2), one should note that the fixed index can be any one of the four.
The Basic Operators

3° Given a tensor $L$ of riemann type, we may form a tensor $S := c(L)$ of ricci type by the following contraction:

$$S_{j\ell} := G^{\ell p} L_{p j i \ell}$$

Let us show that $S$ meets condition (3):

$$S_{\ell j} = G^{\ell p} L_{p \ell \ell j}$$
$$= G^{\ell p} L_{i j p \ell}$$
$$= S_{j\ell}$$

Hence, $S$ is a tensor of ricci type. It may happen that $c(L) = 0$. In that case, one refers to $L$ as a tensor of weyl type.

Given two tensors $S$ and $T$ of ricci type, we may form a tensor $L := S \cdot T$ of riemann type as follows:

$$L_{i j k \ell} := S_{i k} T_{j \ell} + S_{j \ell} T_{i k} - S_{\ell \ell} T_{j k} - S_{j k} T_{i \ell}$$

By routine computation, one can verify that $L$ is a tensor of riemann type. Moreover, it is obvious that:

(4) \[ S \cdot T = T \cdot S \]

Given a tensor $S$ of ricci type, one may introduce the corresponding ricci scalar $s := t(S)$, as follows:

$$s := G^{ij} S_{ij}$$

Finally, for any tensor $S$ of ricci type, we have the following basic relation:

(5) \[ c(G \cdot S) = 2S + sG \]

Let us prove that it is so:

$$(c(G \cdot S))_{j\ell} = G^{\ell p} (G_{pi} S_{ji\ell} + G_{ji\ell} S_{pi} - G_{p\ell} S_{ji} - G_{ji} S_{p\ell})$$
$$= 4S_{j\ell} + G_{ji\ell} t(S) - S_{j\ell} - S_{j\ell}$$
$$= 2S_{j\ell} + sG_{j\ell}$$

In particular:

$$c(G \cdot G) = 6G$$
The Canonical Decomposition

4° Now let $K$ be any tensor of riemann type. It might be the riemann curvature tensor defined by $G$ but it might not. We contend that there exist a tensor $S$ of ricci type and a tensor $W$ of weyl type such that:

\[(*)\quad K = (G \cdot S) + W\]

Moreover, we contend that $S$ and $W$ so described are unique.

To prove these contentions, we simply display the following consequence of relation $(*)$:

\[c(K) = 2S + sG + c(W)\]

Let $R$ stand for $c(K)$. Clearly, $c(W) = 0$ iff:

\[R = 2S + sG\]

which is to say that:

\[S = \frac{1}{2}R - \frac{1}{12}rG\]

where $r := t(R)$. These observations prove both contentions.

Notes

5° Obviously, $R = 0$ iff $K = W$.

6° One says that $G$ is an einstein metric iff there exists a real number $y$ such that $R = yG$. Clearly, that is so iff there exists a real number $z$ such that $S = zG$ iff $6S = R$. The canonical decomposition of $K$ would take the form:

\[K = \frac{1}{6}y(G \cdot G) + W\]

where $W$ is the appropriate tensor of weyl type.

7° One says that $G$ is locally conformally flat iff, for each space-time point $x$, there exist a neighborhood $V$ of $x$ and a positive function $h$ defined on $V$ such that (on $V$) $\bar{G} := hG$ is flat (which is to say that the riemann curvature tensor $\bar{R}$ defined by $\bar{G}$ equals 0. One can prove that $G$ is locally conformally flat iff $W = 0$.

8° One defines the einstein tensor as follows:

\[E := R - \frac{1}{2}rG\]
from which we obtain:

\[ e := t(E) = -r \]

\[ R = E - \frac{1}{2}eG \]

\[ S = \frac{1}{2}E - \frac{1}{6}eG \]

and hence:

\[ K = G \cdot (\frac{1}{2}E - \frac{1}{6}eG) + W \]